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Quantum simulators for Abelian lattice gauge theories

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I want to talk about the possibility
that there is to be an exact simulation,
that the computer will do exactly
the same as nature.

Richard P. Feynman

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Introduction

In 1982 Feynman published an inspiring article entitled *Simulating Physics with Computers* [1], in which he introduced the idea of a universal computer able to simulate physical systems. He began his dissertation by discussing the role of simulation methods, which in his opinion give an important contribution to the comprehension of the laws governing physical phenomena. We usually interpret the word *simulation* as a numerical simulation, performed with an usual computer.

Feynman's idea of simulation developed in the article is quite different. He imagined the universal computer as a real physical system, whose dynamics can be controlled, that evolves in the same way as the phenomenon to be simulated.

He did not interpret, therefore, the simulation as a generation of a succession of states compatible with a certain model, as it happens with numerical simulations, but as the realization of an apparatus which actually evolves like the system we are interested in. Remarkable advantages follow from such an approach: if we perform a numerical (classical) simulation of a quantum system, a linear increase in the degrees of freedom causes an exponential increase of the computational complexity. A quantum simulator is instead an object whose complexity should linearly scale with the size of the physical system we want to simulate.

In the very last years many theoretical studies [2, 3, 4, 5, 6] have been performed about the possibility to implement the dynamics of lattice gauge theories with quantum simulators realized with ultracold atoms trapped in an optical lattice [7]. The reason of the great interest in these theories, and also the attempt to implement them with quantum simulators, is on one hand the importance these theories have in the description of many physical systems: from particle physics to the description of quantum spin liquids, to quantum information theory. On the other hand, numerical simulations of these theories are strongly influenced by the exponential aforementioned growth of the computational complexity.

In some works [2, 3] particular attention is paid to the necessity of imposing suitable constraints on the evolution of the simulator, to limit the unwanted dynamics not predicted by the gauge theory we are simulating. The simplest theory whose implementation can be studied is a $U(1)$ Abelian gauge theory [2], but many efforts have been devoted also to simulation of non Abelian gauge theories [4, 3]. The gauge fields have been defined on the simulator, as functions of its degrees of freedom: some models use internal degrees of freedom of atoms,

while others implement the gauge fields as functions of the number of different atoms on lattice sites.

In this work we study a model for the implementation of an Abelian gauge theory on a one-dimensional lattice. An important step towards the development of models for quantum simulators is represented by the possibility to define discrete gauge fields on the lattice [7], and this is the aspect on which our attention is focused: we study the relation existing between the definition of gauge fields on a discrete Hilbert space and the possible symmetry groups which can be implemented.

Let us remark that when we define discrete variables for the gauge fields, we pass from a statistical approach to lattice gauge theories [8] to a more quantum mechanical one.

The thesis is organized as follows.

In the first chapter Abelian gauge theories are introduced. We focus on the implementation of the minimal coupling between the free Dirac field and the electromagnetic field: in particular, we are interested in the definition of the covariant derivative, which is defined in two ways. The first is more usual, with the introduction of the gauge field [9], while the second consists in a geometrical approach: the covariant derivative is introduced within the definition of the comparator, and the features of this quantity are studied. We then consider the quantization of fields, and focus on the implementation of the gauge transformations by introducing their generators, for the Dirac field and the electromagnetic field respectively. The gauge invariance of the model is required, and from this condition a constraint on physical states is derived.

In the second chapter our goal is to construct a Hamiltonian for a fermion field on a lattice in interaction with an Abelian gauge field. We introduce the problem of fermion doubling, which arises in the space discretization process, and present the solution we adopt, namely the staggered fermions formalism. After the definition of the Hamiltonian for the free-field theory we quantize our model and introduce the interaction with a gauge field, by defining the comparator and the electric field on the lattice.

In the third chapter we study how it is possible to implement the dynamics of the model defined in the preceding chapter by using a quantum simulator consisting of a cloud of ultracold atoms trapped in an optical lattice. The analysis is performed in a number of steps: we first have to represent the degrees of freedom of the theory we want to simulate as observables on the simulator. Then the dynamics of the simulator is studied, in particular its Hamiltonian: we find that, with appropriate initial conditions, the dynamics of the Abelian gauge theory emerges as a second order effective dynamics from that of the quantum simulator. An important formal aspect in this chapter is the introduction of the Quantum Link model to pass from continuous gauge fields to discrete ones.

In the fourth chapter an alternative model to the Quantum Link Model is discussed. A different way to achieve the discretization of the gauge field is then

proposed: it requires the introduction of the continuous Weyl group and the discrete Schwinger-Weyl group. A first contribution to the original part of the thesis is then presented, that is the definition of a discrete radiation field on the lattice by using the formalism of discrete Schwinger-Weyl's operators. We define a new operator for the comparator, and also a new operator for the electric field. The properties of this quantities are explained, and the differences from their Quantum link model analogues are discussed.

The fourth chapter paves the way for the last chapter of this work, in which the major original contribution is presented. We implement a local $\mathbb{Z}_n$ gauge theory on the lattice. By using the commutation properties of the discrete Schwinger-Weyl's operators we define local gauge transformations on the lattice, and explicitly show the gauge invariance of the theory. We find the form of the constraint which physical states must obey, namely Gauss' law. Some differences emerge with respect to the model with continuous gauge fields defined in the second chapter, and we analyze them. Finally, we define a new term for the energy of the electric field as a function of the new operators we have defined; we focus on its properties and compare it with other possible definitions of the same energetic term.

In the Conclusions the results we have achieved are summarized; we also compare all aspects of the theory we have implemented and give a possible outlook for this work.

Chapter 1

Abelian gauge theories

In this chapter Abelian gauge theories are discussed. The free Dirac and electromagnetic fields are introduced; then we promote the free-field theory to a gauge theory, and focus on the definition of covariant derivative. Finally we present the quantization of fields, paying special attention to local gauge transformations and how they act on fields.

1.1 Classical field theory

1.1.1 Notation

The space in which physical phenomena take place is the flat, four dimensional Minkowski space, whose coordinates are

$$x^\mu = (x^0, \mathbf{x}). \tag{1.1.1}$$

The component x^0 is the temporal coordinate and $\mathbf{x}$ is the spatial position vector. Bold letters, like $\mathbf{a}$, will represent three dimensional vectors and the Euclidean scalar product will be denoted $\mathbf{a} \cdot \mathbf{b}$. The metric tensor is

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.1.2}$$

We will adopt the convention to consider Greek indices running over $0, 1, 2, 3$, while Latin indices over $1, 2, 3$; also sum over repeated indices is understood, therefore, for example

$$x^\mu x_\mu = \eta_{\mu\nu} x^\mu x^\nu = (x^0)^2 - |\mathbf{x}|^2. \tag{1.1.3}$$

A generic contravariant and covariant four-vector will be indicated respectively as

$$A^\mu = (A^0, \mathbf{A}) \quad \text{and} \quad A_\mu = (A_0, -\mathbf{A}). \tag{1.1.4}$$

In particular, the four momentum of a particle with mass m and energy E reads

$$p^\mu = (E, \mathbf{p}) \quad \text{with} \quad p^2 = p_\mu p^\mu = m^2, \tag{1.1.5}$$

in which $\mathbf{p}$ is the spatial momentum. Derivatives are written in the form

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \nabla \right). \tag{1.1.6}$$

Let us recall Pauli matrices,

$$\begin{gathered} \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{gathered} \tag{1.1.7}$$

with the shorthand notation $\sigma^\mu = (\mathbf{1}, \sigma^i)$, $\bar{\sigma}^\mu = (\mathbf{1}, -\sigma^i)$ [9]. The Dirac matrices in the chiral representation are:

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ \bar{\sigma}^i & \mathbf{0} \end{pmatrix}, \tag{1.1.8}$$

and according to the slash notation, given a covariant four vector A_μ one defines $\not{A} = \gamma^\mu A_\mu$. Finally, we will set $\hbar = c = e = 1$.

1.1.2 The Dirac field

Coordinates transform under the Lorentz group, that is

$$x^\mu \to \Lambda^\mu{}_\nu \, x^\nu. \tag{1.1.9}$$

The quantity $\Lambda^\mu{}_\nu$ is a 4×4 pseudo-orthogonal matrix of the restricted $SO(3,1)$ with unit determinant and $\Lambda^0{}_0 \geq 0$.

Fields are functions defined on the Minkowski space which obey specific transformation rules under the Lorentz group: Dirac fields form a linear space of functions on which a representation of the Lorentz group is defined. A Lorentz transformation is the composition of a spatial rotation and a boost, and is characterized by six parameters: three characterize the spatial rotation, and three are the rapidities related to the boost. We indicate the first with $\boldsymbol{\theta}$ and the second with $\boldsymbol{\beta}$. The Pauli matrices are generators for two representations of the Lorentz group. Both these representations act on the linear space of two-components fields called spinors, and are therefore called spinorial representations. We call the first representation left handed, the second right handed, and they act on left and right handed spinors respectively. If we indicate with ψ_L and ψ_R a left and a right handed spinor field, the two representations take the form [9]

$$\psi_L \to \Lambda_L \psi_L = \exp\{(-i\boldsymbol{\theta} - \boldsymbol{\beta}) \cdot \boldsymbol{\sigma}/2\} \psi_L, \tag{1.1.10}$$

$$\psi_R \to \Lambda_R \psi_R = \exp\{(-i\boldsymbol{\theta} + \boldsymbol{\beta}) \cdot \boldsymbol{\sigma}/2\} \psi_R. \tag{1.1.11}$$

Left handed and right handed spinor fields describe particles with spin 1/2. They differ for a physical quantity, called helicity. For a plane-wave spinor field with spatial momentum $\mathbf{p}$, helicity is defined as

$$h = \frac{\mathbf{p} \cdot \mathbf{s}}{|\mathbf{p}|}. \tag{1.1.12}$$

It can be found that left-handed spinor fields have $h = -1/2$, while right-handed spinor fields are characterized by $h = 1/2$.

Dirac fields are four-components fields $\Psi(x)$ which describe the evolution of spin-$\frac{1}{2}$ particles. The four degrees of freedom are given by spin and helicity. The dynamics follows from the Lagrangian density of the system, which for a free Dirac field is

$$\mathcal{L}_D = \bar{\Psi}(x)(i\partial\!\!\!/ - m)\Psi(x), \tag{1.1.13}$$

with $\bar{\Psi}(x) = \Psi^\dagger(x)\gamma^0$ and m the mass. In the following, unless necessary, we will not specify that we are talking about densities and we will write Lagrangian or Hamiltonian to indicate Lagrangian density and Hamiltonian density, respectively. The action S is given by

$$S = \int \mathrm{d}t L = \int \mathrm{d}^4x\, \mathcal{L}(\Psi, \partial_\mu \Psi); \tag{1.1.14}$$

by requiring its stationarity with respect to Ψ and its derivatives one obtains the Dirac equation:

$$(i\partial\!\!\!/ - m)\Psi(x) = 0. \tag{1.1.15}$$

We can identify the first two components of the Dirac spinor with the left handed components, and the two others with the right handed ones:

$$\Psi(x) = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \tag{1.1.16}$$

In the chiral basis the Dirac equation splits into two equations:

$$\bar{\sigma}^\mu i\partial_\mu \psi_L = m\psi_R, \tag{1.1.17}$$
$$\sigma^\mu i\partial_\mu \psi_R = m\psi_L. \tag{1.1.18}$$

If we put $m = 0$ we obtain two uncoupled equations whose solutions ψ_L and ψ_R are a massless left and right handed spinor fields, respectively. For $m \neq 0$, plane-wave solutions with positive and negative energy are respectively in the form

$$\Psi(x) = u(p)\mathrm{e}^{-ipx}, \tag{1.1.19}$$
$$\Psi(x) = v(p)\mathrm{e}^{ipx}, \tag{1.1.20}$$

where p is the four momentum, $u(p)$ and $v(p)$ are four-components spinors. The four momentum p is related to the mass m by the relation (1.1.5). A general solution is given by a superposition of plane waves,

$$\Psi(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \sum_{s=1,2} [a_{s,\mathbf{p}} u_s(p) \mathrm{e}^{-ipx} + b^*_{s,\mathbf{p}} v_s(p) \mathrm{e}^{ipx}]\big|_{p^0=E_{\mathbf{p}}}. \tag{1.1.21}$$

In such a superposition positive and negative energy solutions are included, multiplied by coefficients $a_{s,\mathbf{p}}$ and $b_{s,\mathbf{p}}$ respectively; the summation on s runs over solutions with opposite spin components along a reference direction.

1.1.3 The electromagnetic field and the minimal coupling

The electromagnetic field is determined by the four vector potential A_μ, whose first component is the scalar potential, and whose spatial components are the vector potential. The electromagnetic field tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.1.22}$$

The electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ can be identified as the components of the electromagnetic tensor:

$$E^i = -F^{0i}, \quad F^{ij} = -\epsilon^{ijk} B^k. \tag{1.1.23}$$

The electromagnetic tensor is antisymmetric under exchange of indices and invariant under gauge transformations of the potential A_μ: given a smooth function $\phi(x)$, we have

$$A_\mu \to A_\mu - \partial_\mu \phi, \tag{1.1.24}$$
$$F_{\mu\nu} \to \partial_\mu A_\nu - \partial_\mu\partial_\nu\phi - \partial_\nu A_\mu + \partial_\nu\partial_\mu\phi = F_{\mu\nu}. \tag{1.1.25}$$

The Lagrangian and Hamiltonian densities of the free electromagnetic field are [9]

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2), \tag{1.1.26}$$
$$H_{EM} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), \tag{1.1.27}$$

and it follows from (1.1.25) that they are invariant under gauge transformations (1.1.24).

We now want to write a Lagrangian which describes the interaction between the electromagnetic field and the Dirac field. Let us start by recalling the Dirac Lagrangian (1.1.13),

$$\mathcal{L}_D = \bar\Psi(x)(i\partial\!\!\!/ - m)\Psi(x),$$

and observing that it is symmetric under $U(1)$ global transformations of fields: we choose a real number α that does not depend on x and consider

$$\Psi(x) \to \Psi'(x) = \Psi(x)\mathrm{e}^{i\alpha}, \tag{1.1.28}$$
$$\bar{\Psi}(x) \to \bar{\Psi}'(x) = \bar{\Psi}(x)\mathrm{e}^{-i\alpha}. \tag{1.1.29}$$

It follows from the definition of $\Psi(x)$ that $\mathcal{L}(\Psi', \bar{\Psi}') = \mathcal{L}(\Psi, \bar{\Psi})$, therefore we say that the free Dirac field is characterized by a global $U(1)$ symmetry.

The coupling of the Dirac matter field with the electromagnetic field is a consequence of the request of a local symmetry under transformations of $U(1)$. A local $U(1)$ transformation acts on fields Ψ as follows:

$$\Psi(x) \to \Psi'(x) = \Psi(x)\mathrm{e}^{i\alpha(x)}, \tag{1.1.30}$$
$$\bar{\Psi}(x) \to \bar{\Psi}'(x) = \bar{\Psi}(x)\mathrm{e}^{-i\alpha(x)}. \tag{1.1.31}$$

in which $\alpha(x)$ is a real function. In the free Lagrangian (1.1.13) one can observe that the mass term is symmetric also under local transformations, while the kinetic term is not, due to the presence of field derivatives. In order to restore the symmetry, the derivative is replaced by the vector operator D_μ, called the covariant derivative. Let us introduce a vector field, which we identify with the four vector potential A_μ, called the gauge field, in terms of which the covariant derivative is defined as

$$D_\mu = \partial_\mu + iA_\mu. \tag{1.1.32}$$

In the next section the geometrical meaning of the covariant derivative D_μ will be explained. Then, we replace in the Dirac Lagrangian the partial derivative ∂_μ with the covariant derivative:

$$\mathcal{L}_{MC} = \bar{\Psi}(x)(i\not{D} - m)\Psi(x). \tag{1.1.33}$$

This replacement implements the minimal coupling between the electromagnetic field and the Dirac field. Local gauge transformations are defined as follows:

$$\Psi(x) \to \Psi'(x) = \Psi(x)\mathrm{e}^{i\alpha(x)}, \tag{1.1.34}$$
$$D_\mu \to D'_\mu = \partial_\mu + i(A_\mu - \partial_\mu\alpha(x)). \tag{1.1.35}$$

Therefore one has that

$$\begin{aligned}(D_\mu\Psi(x))' &= (\partial_\mu + i(A_\mu - \partial_\mu\alpha(x))(\Psi(x)\mathrm{e}^{i\alpha(x)}) \\ &= \mathrm{e}^{i\alpha(x)}\partial_\mu\Psi(x) + i\Psi(x)\mathrm{e}^{i\alpha(x)}\partial_\mu\alpha(x) \\ &\quad + iA_\mu\Psi(x)\mathrm{e}^{i\alpha(x)} - i\Psi(x)\mathrm{e}^{i\alpha(x)}\partial_\mu\alpha(x) \\ &= \mathrm{e}^{i\alpha(x)}(\partial_\mu + iA_\mu)\Psi(x) \\ &= \mathrm{e}^{i\alpha(x)}D_\mu\Psi(x)\end{aligned} \tag{1.1.36}$$

and finally

$$\bar{\Psi}(x)D_\mu\Psi(x) \xrightarrow{U(1)_{loc}} \bar{\Psi}'(x)D'_\mu\Psi'(x) = \bar{\Psi}(x)D_\mu\Psi(x). \tag{1.1.37}$$

In this way we introduce an interacting term between two fields by imposing the local symmetry of the model under $U(1)$ transformations. The complete Lagrangian, which describes the dynamics of a Dirac field interacting with an electromagnetic field, reads

$$\mathcal{L} = \bar{\Psi}(x)(i\not{D} - m)\Psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1.1.38}$$

1.1.4 Geometrical realization of minimal coupling

In this section an alternative construction of minimal coupling is discussed, since through this procedure we will define a quantity which will be useful in the following. We start by recalling that the implementation of the local symmetry required the definition of a differential operator such that its transformation properties were the same as those of the fields, as seen in (1.1.36). Let us write explicitly the derivative of a Dirac field along the direction identified by the unit vector $\hat{\eta}$:

$$\partial_{\hat{\eta}}\Psi(x) \equiv \lim_{\epsilon\to 0}\frac{\Psi(x+\epsilon\hat{\eta}) - \Psi(x)}{\epsilon} = \hat{\eta}^\mu\partial_\mu\Psi(x). \tag{1.1.39}$$

When we transform $\Psi(x)$ with a local $U(1)$ transformation, $\partial_{\hat{\eta}}\Psi(x)$ does not transform in the same way as $\Psi(x)$: according to (1.1.39) we have

$$(\partial_{\hat{\eta}}\Psi(x))' = \lim_{\epsilon\to 0}\frac{\mathrm{e}^{i\alpha(x+\epsilon\hat{\eta})}\Psi(x+\epsilon\hat{\eta}) - \mathrm{e}^{i\alpha(x)}\Psi(x)}{\epsilon}. \tag{1.1.40}$$

Now we introduce a quantity indicated $U(x,y)$, called the comparator, which transforms under local $U(1)$ operations as [7]:

$$U(x,y) \to \mathrm{e}^{i\alpha(x)}U(x,y)\mathrm{e}^{-i\alpha(y)}. \tag{1.1.41}$$

We infer that

$$\begin{aligned} U(x,y)\Psi(y) \to& \mathrm{e}^{i\alpha(x)}U(x,y)\mathrm{e}^{-i\alpha(y)}\mathrm{e}^{i\alpha(y)}\Psi(y) \\ &= \mathrm{e}^{i\alpha(x)}U(x,y)\Psi(y). \end{aligned} \tag{1.1.42}$$

Let us define the quantity [10]

$$D_{\hat{\eta}}\Psi(x) = \lim_{\epsilon\to 0}\frac{U(x,x+\epsilon\hat{\eta})\Psi(x+\epsilon\hat{\eta}) - \Psi(x)}{\epsilon}, \tag{1.1.43}$$

which we call covariant derivative; in particular, by choosing a unit vector $\hat{\mu}$ aligned with a reference axis of the Minkowski space, one gets

$$D_\mu\Psi(x) = \lim_{\epsilon\to 0}\frac{U(x,x+\epsilon\hat{\mu})\Psi(x+\epsilon\hat{\mu}) - \Psi(x)}{\epsilon}. \tag{1.1.44}$$

A first observation is that $D_{\hat{\eta}}\Psi(x)$ defined in (1.1.43) transforms like $\Psi(x)$, since

$$\begin{aligned}(D_{\hat{\eta}}\Psi(x))' =& \lim_{\epsilon\to 0}\frac{\mathrm{e}^{i\alpha(x)}(U(x,x+\epsilon\hat{\eta})\Psi(x+\epsilon\hat{\eta})-\Psi(x))}{\epsilon}\\ =&\mathrm{e}^{i\alpha(x)}\lim_{\epsilon\to 0}\frac{U(x,x+\epsilon\hat{\eta})\Psi(x+\epsilon\hat{\eta})-\Psi(x)}{\epsilon}\\ =&\mathrm{e}^{i\alpha(x)}D_{\hat{\eta}}\Psi(x),\end{aligned} \tag{1.1.45}$$

therefore it is a gauge-covariant quantity. Now we have to show that the covariant directional derivative D_μ coincides with the covariant derivative defined in (1.1.32) with the gauge field A_μ. First, let us assume that $U(x,y)$ is unitary, therefore there exists a function, $\phi(x,y)$, such that $U(x,y)=\mathrm{e}^{i\phi(x,y)}$, and let us impose that $U(x,x)=1$, so $U^{-1}(x,y)=U(y,x)$. If $\phi(x,y)$ is regular we can consider its derivatives with respect to the second argument and call them $\partial_\mu\phi(x,y)=\tilde{A}_\mu$. Therefore a first order approximation in ϵ of $U(x,x+\hat{\eta}\epsilon)$ reads

$$U(x,x+\epsilon\hat{\eta})\simeq 1+i\epsilon\hat{\eta}^\mu\tilde{A}_\mu. \tag{1.1.46}$$

The covariant derivative can be written according to (1.1.46) as follows:

$$\begin{aligned}D_{\hat{\eta}}\Psi(x) =& \lim_{\epsilon\to 0}\frac{(1+i\epsilon\hat{\eta}^\mu\tilde{A}_\mu)\Psi(x+\epsilon\hat{\eta})-\Psi(x)}{\epsilon}\\ =&\lim_{\epsilon\to 0}\frac{\Psi(x+\epsilon\hat{\eta})-\Psi(x)+i\epsilon\hat{\eta}^\mu\tilde{A}_\mu\Psi(x+\epsilon\hat{\eta})}{\epsilon}\\ =&\hat{\eta}^\mu(\partial_\mu+i\tilde{A}_\mu)\Psi(x),\end{aligned} \tag{1.1.47}$$

therefore the two definitions (1.1.32) and (1.1.44) for the covariant derivative are equivalent. With the introduction of the field $\tilde{A}_\mu$ the form of $U(x,y)$ is [7]

$$U(x,y)=\exp\left\{i\int_x^y \mathrm{d}x^\mu\tilde{A}_\mu\right\}. \tag{1.1.48}$$

Now we are interested in the transformation properties of the vector field $\tilde{A}_\mu$: we apply the transformation rule (1.1.41) to $U(x,x+\epsilon\hat{\eta})$, and neglect second order terms, obtaining

$$\begin{aligned}U(x,x+\epsilon\hat{\eta})\to&\,\mathrm{e}^{i\alpha(x)}U(x,x+\epsilon\hat{\eta})\mathrm{e}^{-i\alpha(x+\epsilon\hat{\eta})}\\ \simeq&\,\mathrm{e}^{i\alpha(x)}(1+i\epsilon\hat{\eta}^\mu\tilde{A}_\mu)(1-i\epsilon\hat{\eta}^\mu\partial_\mu\alpha(x))\mathrm{e}^{-i\alpha(x)}\\ =&\,1+i\epsilon\hat{\eta}^\mu(\tilde{A}_\mu-\partial_\mu\alpha(x))+O(\epsilon^2),\end{aligned} \tag{1.1.49}$$

from which it follows that transforming the comparator with (1.1.41) is equivalent to transforming the field A_μ with (1.1.24). Therefore we can identify the field that appears in (1.1.48) with the four vector potential in (1.1.32); henceforth we will identify the gauge field and the field in (1.1.48) and we will indicate it with A_μ.

Now we want to study a gauge invariant quantity involving the comparator: a first observation is that, by construction, $U(x,x)$ is gauge invariant, and that a product of comparators in the form

$$U(x_1,x_2)U(x_2,x_3)\dots U(x_{n-1},x_n)$$

transforms like $U(x_1,x_n)$. Let us take two directions in the Minkowski space and the correspondent unit vectors: for example, we choose

$$\hat{\eta}_1 = (0,1,0,0) \tag{1.1.50}$$

$$\hat{\eta}_2 = (0,0,1,0). \tag{1.1.51}$$

The following quantity is gauge invariant:

$$\begin{aligned}\mathbf{U}(x) \equiv & U(x,x+\epsilon\hat{\eta}_1)U(x+\epsilon\hat{\eta}_1,x+\epsilon\hat{\eta}_1+\epsilon\hat{\eta}_2)\\ & \times U(x+\epsilon\hat{\eta}_2+\epsilon\hat{\eta}_1,x+\epsilon\hat{\eta}_2)U(x+\epsilon\hat{\eta}_2,x),\end{aligned} \tag{1.1.52}$$

in which ϵ is an infinitesimal parameter. The quantity $\mathbf{U}(x)$ is the comparator calculated along the square closed path with side length ϵ, starting from the point x, in the plane identified by the vectors $\hat{\eta}_1$ and $\hat{\eta}_2$: this square is usually called plaquette, therefore $\mathbf{U}(x)$ is the comparator calculated along a plaquette. To perform the computation we need to consider a second order term for $U(x,x+\epsilon\hat{\eta})$ [10]:

$$U(x,x+\epsilon\hat{\eta}) = \exp\{i\epsilon\hat{\eta}^\mu A_\mu(x+\frac{1}{2}\epsilon\hat{\eta})+O(\epsilon^3)\}. \tag{1.1.53}$$

This term is the only for which we have that

$$\begin{aligned}U(x,x+\epsilon\hat{\eta})^{-1} &= U(x+\epsilon\hat{\eta},x)\\ &= \exp\{-i\epsilon\hat{\eta}^\mu A_\mu(x+\epsilon\hat{\eta}-\frac{1}{2}\epsilon\hat{\eta})+O(\epsilon^3)\}\\ &= \exp\{-i\epsilon\hat{\eta}^\mu A_\mu(x+\frac{1}{2}\epsilon\hat{\eta})+O(\epsilon^3)\}\\ &= U(x,x+\epsilon\hat{\eta})^*,\end{aligned} \tag{1.1.54}$$

and therefore the requirement of unitarity of the comparator can be satisfied without introducing additional second order terms. Within approximation (1.1.53), $\mathbf{U}(x)$ can be written as

$$\begin{aligned}\mathbf{U}(x) &= \exp\{i\epsilon A_1(x+\frac{1}{2}\epsilon\hat{\eta}_1)i\epsilon A_2(x+\epsilon\hat{\eta}_1+\frac{1}{2}\epsilon\hat{\eta}_2)\\ &\quad - i\epsilon A_1(x+\epsilon\hat{\eta}_2+\frac{1}{2}\epsilon\hat{\eta}_1) - i\epsilon A_2(x+\frac{1}{2}\epsilon\hat{\eta}_2)+O(\epsilon^3)\}\\ &= \exp\{i\epsilon[A_1(x)+\frac{\epsilon}{2}\partial_1 A_1(x)+A_2(x)+\epsilon\partial_1 A_2(x)+\frac{\epsilon}{2}\partial_2 A_2(x)\\ &\quad - A_1(x)-\epsilon\partial_2 A_1(x)-\frac{\epsilon}{2}\partial_1 A_1(x)-A_2(x)-\frac{\epsilon}{2}\partial_2 A_2(x)]+O(\epsilon^3)\}\\ &= \exp\{i\epsilon^2[\partial_1 A_2(x)-\partial_2 A_1(x)]+O(\epsilon^3)\}\\ &= \exp\{i\epsilon^2[(\nabla\wedge\mathbf{A})_3]+O(\epsilon^3)\}.\end{aligned} \tag{1.1.55}$$

Since $\mathbf{U}(x)$ is gauge invariant, the term $\partial_1 A_2(x) - \partial_2 A_1(x)$ must be gauge invariant too, for all couple of directions in Minkowski space. It follows that we can define a gauge invariant tensor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$: it coincides with the gauge invariant electromagnetic tensor (1.1.22). In the above calculation, the gauge invariance has been proved in an alternative way with respect to (1.1.25). A final observation is about the physical meaning of $\mathbf{U}(x)$, when the calculation (1.1.55) is performed along two spatial directions: by (1.1.55) we see that the exponent is the third component of the curl of A_μ, multiplied by ϵ^2, namely the flux of the third component of the magnetic field across a surface with area ϵ^2, in the plane generated by unit vectors $(\hat{\eta}_1, \hat{\eta}_2)$.

1.2 Field quantization

1.2.1 Quantization of the Dirac Field

Given the Dirac field $\Psi(x)$ with the Lagrangian density $\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi$, its canonically conjugate momentum is defined as follows:

$$\Pi_\Psi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi(x))} = i\Psi^\dagger(x). \tag{1.2.1}$$

The canonical quantization of fields consists in promoting the classical fields $\Psi(x)$ and $\Pi_\Psi(x)$ to field operators $\hat{\Psi}(x)$ and $\hat{\Pi}_\Psi(x)$ whose components satisfy the equal-time anticommutation relations [9]

$$\begin{aligned} &- i\{\hat{\Psi}_a(x^0, \mathbf{x}), \hat{\Pi}_{\Psi,\, b}(x^0, \mathbf{x}')\} \\ &= \{\hat{\Psi}_a(x^0, \mathbf{x}), \hat{\Psi}_b^\dagger(x^0, \mathbf{x}')\} = \delta^{(3)}(\mathbf{x} - \mathbf{x}')\delta_{a,b}. \end{aligned} \tag{1.2.2}$$

By recalling the wave expansion (1.1.21) of the field $\Psi(x)$,

$$\Psi(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3\sqrt{E_\mathbf{p}}} \sum_{s=1,2} [a_{s,\mathbf{p}} u_s(p) \mathrm{e}^{-ipx} + b^*_{s,\mathbf{p}} v_s(p) \mathrm{e}^{ipx}]\big|_{p^0 = E_\mathbf{p}},$$

the quantization procedure is equivalent to replace the coefficients $a_{\mathbf{p},s}$ and $b_{\mathbf{p},s}$ with the operators $\hat{a}_{\mathbf{p},s}$ and $\hat{b}_{\mathbf{p},s}$ which obey the following anticommutation relations

$$\{\hat{a}_{\mathbf{p},s}, \hat{a}^\dagger_{\mathbf{q},r}\} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})\delta_{s,r}, \tag{1.2.3}$$

$$\{\hat{b}_{\mathbf{p},s}, \hat{b}^\dagger_{\mathbf{q},r}\} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})\delta_{s,r}, \tag{1.2.4}$$

while all other anticommutators vanish.

The Hilbert space $\mathcal{H}$ on which these operators act has the structure of a Fock space, namely it is the direct sum of many Hilbert spaces, one for each combination of a spin and a momentum value. The states of these Hilbert space

are characterized by the numbers of particles with a certain momentum and spin. We define a vacuum vector in $\mathcal{H}$, $|0\rangle$, such that

$$\hat{a}_{\mathbf{p},s}|0\rangle = 0, \tag{1.2.5}$$

$$\hat{b}_{\mathbf{p},s}|0\rangle = 0, \qquad \forall \mathbf{p}, s. \tag{1.2.6}$$

Operators $\hat{a}_{\mathbf{p},s}$ refers to particles, $\hat{b}_{\mathbf{p},s}$ to antiparticles. By considering operators $\hat{a}^\dagger_{\mathbf{p},s}$ and $\hat{b}^\dagger_{\mathbf{p},s}$ we have that

$$\sqrt{2E_{\mathbf{p}}}\hat{a}^\dagger_{\mathbf{p},s}|0\rangle = |\mathbf{p}, s\rangle, \tag{1.2.7}$$

$$\sqrt{2E_{\mathbf{p}}}\hat{b}^\dagger_{\mathbf{p},s}|0\rangle = |\mathbf{p}^*, s\rangle, \qquad \forall \mathbf{p}, s \tag{1.2.8}$$

in which the * indicates that we are referring to antiparticles. Therefore $\hat{a}^\dagger_{\mathbf{p},s}$ and $\hat{b}^\dagger_{\mathbf{p},s}$ act as creation operators of a particle and an antiparticle with momentum $\mathbf{p}$ and spin s. We can construct many-particle states by applying creation operators relative to different momentum and spin values. Due to the anticommutation relations obeyed by Dirac fields a state cannot contain more than one particle with the same momentum, spin and electric charge. A state with n particles, constructed by repeated applications of $\hat{a}^\dagger$ creation operators, reads

$$\begin{aligned} |\Phi\rangle &= |\mathbf{p}_1, s_1; \ldots; \mathbf{p}_n, s_n\rangle \\ &= (2E_{\mathbf{p}_1} \ldots 2E_{\mathbf{p}_n})^{1/2} \hat{a}^\dagger_{\mathbf{p}_1,s_1} \ldots \hat{a}^\dagger_{\mathbf{p}_n,s_n}|0\rangle; \end{aligned} \tag{1.2.9}$$

in this state one particle has momentum $\mathbf{p}_1$ and spin s_1, and so on. In the same way a state including antiparticle could be obtained by applying $\hat{b}^\dagger$ operators. Operators $\hat{a}_{\mathbf{p},s}$ and $\hat{b}_{\mathbf{p},s}$ destroy a particle and an antiparticle respectively with momentum $\mathbf{p}$ and spin s, if the state contains such a particle, otherwise they annihilate it.

The Hamiltonian density operator is obtained from the classical theory by replacing the classical fields with the correspondent field operators. In this case

$$\hat{H} = \hat{\Pi}_\Psi \hat{\Psi} - \mathcal{L} = \hat{\bar{\Psi}}(-i\gamma^i \partial_i + m)\hat{\Psi}. \tag{1.2.10}$$

Now let us take a look to symmetries of this system. When dealing with classical fields we observe that the free Dirac field is symmetric under global $U(1)$ transformations. Also, it can be shown that the quantity $\Psi^\dagger(x)\Psi(x)$ represents the electric charge density. When we quantize fields, the operator $\hat{\Psi}^\dagger(x)\hat{\Psi}(x)$ is the generator for global and local $U(1)$ transformations of fields. Given a real function $\alpha(x)$, we can define the operator

$$T = \exp\left\{ i \int \mathrm{d}^3\mathbf{x}\, \alpha(x) \hat{\Psi}^\dagger(t, \mathbf{x}) \hat{\Psi}(t, \mathbf{x}) \right\}. \tag{1.2.11}$$

Henceforth, if not explicitly specified, we will assume that all field operators are considered at equal times, so anticommutation rules (1.2.2) can be applied; also,

indices referring to spinors components will be omitted. The field transformation reads:

$$\hat{\Psi}(y) \to T^\dagger \hat{\Psi}(y) T = \hat{\Psi}(y) \mathrm{e}^{i\alpha(y)}. \tag{1.2.12}$$

To demonstrate this result, let us first compute the commutator

$$\begin{aligned}
&\left[\int \mathrm{d}^3\mathbf{x}\, \alpha(x) \hat{\Psi}^\dagger(t,\mathbf{x})\hat{\Psi}(t,\mathbf{x}), \hat{\Psi}(t,\mathbf{y})\right] \\
&= \int \mathrm{d}^3\mathbf{x}\, \alpha(x) \left[\hat{\Psi}^\dagger(t,\mathbf{x})\hat{\Psi}(t,\mathbf{x}), \hat{\Psi}(t,\mathbf{y})\right] \\
&= \int \mathrm{d}^3\mathbf{x}\, \alpha(x) \Big(\hat{\Psi}^\dagger(t,\mathbf{x})\hat{\Psi}(t,\mathbf{x})\hat{\Psi}(t,\mathbf{y}) - \hat{\Psi}(t,\mathbf{y})\hat{\Psi}^\dagger(t,\mathbf{x})\hat{\Psi}(t,\mathbf{x})\Big) \\
&= \int \mathrm{d}^3\mathbf{x}\, \alpha(x) \Big(\hat{\Psi}^\dagger(t,\mathbf{x})\hat{\Psi}(t,\mathbf{x})\hat{\Psi}(t,\mathbf{y}) + \hat{\Psi}^\dagger(t,\mathbf{x})\hat{\Psi}(t,\mathbf{y})\hat{\Psi}(t,\mathbf{x}) \\
&\quad - \delta^{(3)}(\mathbf{x}-\mathbf{y})\hat{\Psi}(t,\mathbf{x})\Big) \\
&= -\alpha(y)\hat{\Psi}(t,\mathbf{y}).
\end{aligned} \tag{1.2.13}$$

Now let us recall the following result from Baker-Campbell-Hausdorff formula: given two operators X and Y, and a real c-number c, such that $[X,Y] = cY$, the following relation holds:

$$\mathrm{e}^X Y \mathrm{e}^{-X} = \mathrm{e}^c\, Y. \tag{1.2.14}$$

Finally, from (1.2.13) we can write

$$\begin{aligned}
&T^\dagger \hat{\Psi}(y) T = \\
&\exp\left\{-i\int \mathrm{d}^3\mathbf{x}\, \alpha(x)\hat{\Psi}^\dagger(x)\hat{\Psi}(x)\right\}\hat{\Psi}(y)\exp\left\{i\int \mathrm{d}^3\mathbf{x}\, \alpha(x)\hat{\Psi}^\dagger(x)\hat{\Psi}(x)\right\} \\
&= \mathrm{e}^{i\alpha(y)}\hat{\Psi}(y),
\end{aligned} \tag{1.2.15}$$

which proves Eq. (1.2.11). The free field Hamiltonian (1.2.10) is invariant only for transformations (1.2.12) performed with a constant function α independent from x, that are global transformations. In this case the generator is not the electric charge density operator, but the electric charge itself, since the operator T takes the form

$$T = \exp\left\{i\alpha\int \mathrm{d}^3\mathbf{x}\, \hat{\Psi}^\dagger(t,\mathbf{x})\hat{\Psi}(t,\mathbf{x})\right\}. \tag{1.2.16}$$

To implement the local symmetry in the quantized theory one has to define the covariant derivative, which is introduced in the last section of this chapter.

1.2.2 Quantization of the electromagnetic field

We now discuss the canonical quantization of the electromagnetic field [11]. We have seen in (1.1.25) that the Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is symmetric for local transformations (1.1.24) of the vector potential. Choosing a specific function $\phi(x)$ to perform a transformation (1.1.24) is equivalent to fixing a gauge. To realize the canonical quantization let us first recall that the dynamical variables of the system described by $\mathcal{L}$ are the components of the four vector potential A_μ, and their conjugated momenta are defined by the relation

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)}. \tag{1.2.17}$$

Now, the field quantization consists in promoting the dynamical variables and their momenta from functions to field operators, and imposing appropriate commutation relations between them. In the specific case of the electromagnetic field, before proceeding in this way let us observe that the momentum Π_0 is zero, since $\partial_0 A_0$ does not appear in $\mathcal{L}$. Therefore any commutation relation involving the temporal component of the four-vector potential cannot be imposed.

The solution we adopt to realize the canonical quantization is to fix a particular gauge: we take a function $\phi(x)$ such that the condition

$$\partial_0 \phi = A_0 \tag{1.2.18}$$

holds, and we perform a gauge transformation with this function $\phi(x)$. It follows indeed that

$$A'_\mu = A_\mu - \partial_\mu \phi(x), \qquad A'_0 = 0. \tag{1.2.19}$$

Henceforth we will consider the four vector potential within this gauge choice, in which $A_0 = 0$. This gauge condition does not completely fix the four vector potential, since we are still free to perform gauge transformations (1.1.24) with space-dependent functions $\phi(\mathbf{x})$.

In the chosen gauge the non vanishing momenta are

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} = -E_i. \tag{1.2.20}$$

We now implement the quantization, by promoting A_i and Π_i to field operators $\hat{A}_i$ and $\hat{\Pi}_i$ which obey the following equal time commutation relation:

$$[\hat{A}_i(t,\mathbf{x}), \hat{\Pi}_j(t,\mathbf{x}')] = i\delta(\mathbf{x}-\mathbf{x}')\delta_{ij}, \tag{1.2.21}$$

namely

$$[\hat{A}_i(t,\mathbf{x}), \hat{E}_j(t,\mathbf{x}')] = -i\delta(\mathbf{x}-\mathbf{x}')\delta_{ij}. \tag{1.2.22}$$

We will now show that the operator $\hat{Q}(\mathbf{x}) = \nabla \cdot \hat{\mathbf{E}}(\mathbf{x})$ is the generator of the gauge transformations of $\hat{A}_i(t,\mathbf{x})$ [11].

Let us first compute the commutator

$$\begin{aligned}
&\left[\int \mathrm{d}\mathbf{z}\,\phi(\mathbf{z})\hat{Q}(\mathbf{z}), \hat{A}_i(t,\mathbf{x})\right] \\
&= \left[\int \mathrm{d}\mathbf{z}\,\phi(\mathbf{z})\nabla\cdot\hat{\mathbf{E}}(\mathbf{z}), \hat{A}_i(t,\mathbf{x})\right] \\
&= -\left[\int \mathrm{d}\mathbf{z}\,\nabla\phi(\mathbf{z})\cdot\hat{\mathbf{E}}(t,\mathbf{z}), \hat{A}_i(t,\mathbf{x})\right] \\
&= \int \mathrm{d}\mathbf{z}\,\nabla^j\phi(\mathbf{z})\left[\hat{E}_j(t,\mathbf{z}), \hat{A}_i(t,\mathbf{x})\right] \\
&= i\int \mathrm{d}\mathbf{z}\,\nabla^j\phi(\mathbf{z})\delta(\mathbf{z}-\mathbf{x})\delta_{ij} = i\int \mathrm{d}z^i\,\nabla^i\phi(\mathbf{z})\delta(z^i-x^i) \\
&= i\,\nabla^i\phi(\mathbf{x}).
\end{aligned} \tag{1.2.23}$$

This result suggests us to implement gauge transformations

$$\hat{A}_i(t,\mathbf{x}) \to \hat{A}_i(t,\mathbf{x}) - \partial_i\phi(\mathbf{x}) \equiv \hat{A}_i(t,\mathbf{x}) - \nabla^i\phi(\mathbf{x}) \tag{1.2.24}$$

by defining the following operator:

$$\begin{aligned}
\hat{W}[\phi] &\equiv \exp\left\{-i\int \mathrm{d}\mathbf{z}\,\phi(\mathbf{z})\hat{Q}(\mathbf{z})\right\} \\
&= \exp\left\{-i\int \mathrm{d}\mathbf{z}\,\phi(\mathbf{z})\nabla\cdot\hat{\mathbf{E}}(t,\mathbf{z})\right\} \\
&= \exp\left\{i\int \mathrm{d}\mathbf{z}\,\nabla\phi(\mathbf{z})\cdot\hat{\mathbf{E}}(t,\mathbf{z})\right\} \\
&= \exp\left\{-i\int \mathrm{d}\mathbf{z}\,\nabla^j\phi(\mathbf{z})\hat{E}_j(t,\mathbf{z})\right\}.
\end{aligned} \tag{1.2.25}$$

The transformation is

$$\hat{A}_i(t,\mathbf{x}) \to \hat{W}^\dagger[\phi]\hat{A}_i(t,\mathbf{x})\hat{W}[\phi], \tag{1.2.26}$$

and we will show that it yields the same result as in (1.2.24): this is easily shown if a particular representation of the field operators is adopted. First, let us consider the Hilbert space associated to the electromagnetic field. We choose as a basis the set $|\{E_i(t,\mathbf{x})\}\rangle$, in which each vector is the electric field value at each position, at a fixed time. The inner product, given two such states $|\{E_i(t,\mathbf{x})\}\rangle$ and $|\{E'_i(t,\mathbf{x})\}\rangle$ reads: A state of the Hilbert space $|\Phi\rangle$ can be projected onto this basis, generating a wave function $\Phi(\{E_i(t,\mathbf{x})\})$ which is a functional of the electric field values. The operator $\hat{E}_i(t,\mathbf{x})$ acts on the wave function as a multiplication one, while $\hat{A}_i(t,\mathbf{x})$ acts as

$$\hat{A}_i(t,\mathbf{x}) = -i\frac{\delta}{\delta\hat{E}_i(t,\mathbf{x})}. \tag{1.2.27}$$

In this representation we have that

$$
\begin{aligned}
&\hat{W}^\dagger[\phi(\mathbf{z})]\hat{A}_i(t,\mathbf{x})\hat{W}[\phi(\mathbf{z})] \\
&= \exp\left\{ i \int \mathrm{d}\mathbf{z}\, \nabla^j \phi(\mathbf{z}) \hat{E}_j(t,\mathbf{z}) \right\} \hat{A}_i(t,\mathbf{x}) \exp\left\{ -i \int \mathrm{d}\mathbf{z}\, \nabla^j \phi(\mathbf{z}) \hat{E}_j(t,\mathbf{z}) \right\} \\
&= \exp\left\{ i \int \mathrm{d}\mathbf{z}\, \nabla^j \phi(\mathbf{z}) \hat{E}_j(t,\mathbf{z}) \right\} \left(-i \frac{\delta}{\delta \hat{E}_i(t,\mathbf{x})} \right) \\
&\quad \times \exp\left\{ -i \int \mathrm{d}\mathbf{z}\, \nabla^j \phi(\mathbf{z}) \hat{E}_j(t,\mathbf{z}) \right\} \\
&= \exp\left\{ i \int \mathrm{d}\mathbf{z}\, \nabla^j \phi(\mathbf{z}) \hat{E}_j(t,\mathbf{z}) \right\} \exp\left\{ -i \int \mathrm{d}\mathbf{z}\, \nabla^j \phi(\mathbf{z}) \hat{E}_j(t,\mathbf{z}) \right\} \\
&\quad \times \left[\left(-i \frac{\delta}{\delta \hat{E}_i(t,\mathbf{x})} (-i) \int \mathrm{d}\mathbf{z}\, \nabla^j \phi(\mathbf{z}) \hat{E}_j(t,\mathbf{z}) \right) - i \frac{\delta}{\delta \hat{E}_i(t,\mathbf{x})} \right] \\
&= -\int \mathrm{d}\mathbf{z}\, \nabla^j \phi(\mathbf{z}) \frac{\delta}{\delta \hat{E}_i(t,\mathbf{x})} \hat{E}_j(t,\mathbf{z}) + \hat{A}_i(t,\mathbf{x}) \\
&= -\int \mathrm{d}\mathbf{z}\, \nabla^j \phi(\mathbf{z}) \delta(\mathbf{x}-\mathbf{z}) \delta_{i\,j} + \hat{A}_i(t,\mathbf{x}) \\
&= -\nabla^i \phi(\mathbf{x}) + \hat{A}_i(t,\mathbf{x}) = \hat{A}_i(t,\mathbf{x}) - \partial_i \phi(\mathbf{x}),
\end{aligned}
\tag{1.2.28}
$$

therefore the operation (1.2.26) implements the gauge transformation defined in (1.2.24).

1.2.3 Gauss' law and gauge transformations

Since gauge transformations are not relevant for the physics of the system, physical states must be invariant under gauge transformations [11]. For a free electromagnetic field, given a state $|\Phi\rangle$ the following local condition must therefore hold:

$$
\hat{W}[\phi(\mathbf{z})]|\Phi\rangle = |\Phi\rangle; \tag{1.2.29}
$$

it is equivalent to

$$
\nabla \cdot \hat{\mathbf{E}}(t,\mathbf{z})|\Phi\rangle = 0 \qquad \forall\, t, \mathbf{z}. \tag{1.2.30}
$$

This is the second quantization counterpart of the classical Maxwell equation

$$
\nabla \cdot \mathbf{E}(t,\mathbf{z}) = 0, \tag{1.2.31}
$$

and is now a constraint which selects physical states by requesting their gauge invariance.

If we now consider a system in which the Dirac and the electromagnetic fields interact with each other, the Hilbert space will be the tensor product of the Hilbert spaces on which the two fields act: physical states like $|\Omega\rangle_{Dirac}|\Phi\rangle_{electr}$ must be invariant for transformations in the form

$$
|\Omega\rangle_{Dirac}|\Phi\rangle_{electr} \to \left(T \otimes \hat{W}[\phi(\mathbf{z})] \right) |\Omega\rangle_{Dirac}|\Phi\rangle_{electr}, \tag{1.2.32}
$$

and this condition equivalently reads

$$(\nabla \cdot \hat{\mathbf{E}}(x) - \hat{\Psi}^\dagger \hat{\Psi}(x))|\Omega\rangle_{Dirac}|\Phi\rangle_{electr} = 0. \tag{1.2.33}$$

This equation is the equivalent of the classical Gauss' law in presence of free charges,

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \rho(\mathbf{x}). \tag{1.2.34}$$

Now, let us define the comparator in the quantized theory:

$$\begin{aligned} \hat{U}(t;\mathbf{x},\mathbf{y}) &\equiv \exp\left\{ -i \int_{\mathbf{x}}^{\mathbf{y}} \mathrm{d}\mathbf{z} \cdot \hat{\mathbf{A}}(t,\mathbf{z}) \right\} \\ &= \exp\left\{ i \int_{\mathbf{x}}^{\mathbf{y}} \mathrm{d}z^i \hat{A}_i(t,\mathbf{z}) \right\}. \end{aligned} \tag{1.2.35}$$

The comparator transforms according to (1.2.28):

$$\begin{aligned} &\hat{U}(t;\mathbf{x},\mathbf{y}) \to \hat{W}^\dagger[\phi(\mathbf{z})]\hat{U}(t;\mathbf{x},\mathbf{y})\hat{W}[\phi(\mathbf{z})] \\ &= \exp\left\{ i \int \mathrm{d}\mathbf{z}\, \nabla^j \phi(\mathbf{z}) \hat{E}_j(t,\mathbf{z}) \right\} \exp\left\{ i \int_{\mathbf{x}}^{\mathbf{y}} \mathrm{d}z^i \hat{A}_i(t,\mathbf{z}) \right\} \\ &\quad \times \exp\left\{ -i \int \mathrm{d}\mathbf{z}\, \nabla^j \phi(\mathbf{z}) \hat{E}_j(t,\mathbf{z}) \right\} \\ &= \exp\left\{ i \int \mathrm{d}\mathbf{z}\, \nabla^j \phi(\mathbf{z}) \hat{E}_j(t,\mathbf{z}) \right\} \exp\left\{ i \int_{\mathbf{x}}^{\mathbf{y}} \mathrm{d}z^i \left(-i \frac{\delta}{\delta \hat{E}_i(t,\mathbf{x})} \right) \right\} \\ &\quad \times \exp\left\{ -i \int \mathrm{d}\mathbf{z}\, \nabla^j \phi(\mathbf{z}) \hat{E}_j(t,\mathbf{z}) \right\} \\ &= \exp\left\{ i \int_{\mathbf{x}}^{\mathbf{y}} \mathrm{d}z^i \left(\hat{A}_i(t,\mathbf{z}) - \partial_i \phi(\mathbf{z}) \right) \right\} \\ &= \mathrm{e}^{i\phi(\mathbf{x})} \hat{U}(t;\mathbf{x},\mathbf{y}) \mathrm{e}^{-i\phi(\mathbf{y})}. \end{aligned} \tag{1.2.36}$$

Note that we have recovered again the transformation rule (1.1.41) for the classical comparator. Let us write now the covariant derivative using the comparator: by recalling the definition (1.1.43) for the classical case we define

$$\hat{D}_{\hat{\eta}} \hat{\Psi}(x) = \lim_{\epsilon \to 0} \frac{\hat{U}(x, x+\epsilon\hat{\eta}) \hat{\Psi}(x+\epsilon\hat{\eta}) - \hat{\Psi}(x)}{\epsilon}, \tag{1.2.37}$$

therefore we obtain the derivative $\hat{D}_\mu$ by choosing the unit vector aligned along the μ axis in the Minkowski space. The Lagrangian for the interaction theory is

$$\hat{\mathcal{L}} = \hat{\bar{\Psi}}^\dagger (i \hat{\not{D}} - m) \hat{\Psi} - \frac{1}{4} \hat{F}_{\mu\nu} F^{\mu\nu}, \tag{1.2.38}$$

with $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu$ and $\hat{\not{D}} = \gamma^\mu \hat{D}_\mu$, while the Hamiltonian is

$$\hat{H} = \hat{\bar{\Psi}}^\dagger (-i\gamma^i \hat{D}_i + m) \hat{\Psi} + \frac{1}{2} (\hat{\mathbf{E}}^2 + \hat{\mathbf{B}}^2), \tag{1.2.39}$$

They are invariant under gauge transformations in the form

$$\hat{\mathcal{L}} \to (T^\dagger \otimes \hat{W}^\dagger[\phi(\mathbf{z})])\, \hat{\mathcal{L}}\, (T \otimes \hat{W}[\phi(\mathbf{z})]). \tag{1.2.40}$$

In particular, the kinetic term for the Dirac field $\hat{\bar{\Psi}}^\dagger \not{D} \hat{\Psi}$ in invariant due to the combined action of the comparator and the Dirac field transformations (1.2.28) and (1.2.12).

Chapter 2

Abelian lattice gauge theory

In this chapter we define the lattice Hamiltonian of a $1+1$ dimensional system with a fermion matter field coupled with an Abelian gauge field. We introduce the problem of fermion doubling and adopt the solution provided by the use of staggered fermions, which will be used in the following chapters. Finally, gauge transformations are defined in this model and the gauge invariance of the Hamiltonian is verified.

2.1 The discrete free Dirac field Hamiltonian

2.1.1 The fermion doubling

The necessity to define staggered fermions is due to the fermion doubling problem, which is introduced with an example [12, 13]. Let us consider the following eigenvalue equation:

$$-i\frac{\mathrm{d}}{\mathrm{d}x}f(x) = \lambda f(x), \tag{2.1.1}$$

with λ eigenvalue of $f(x)$ for the operator $-i\mathrm{d}/\mathrm{d}x$. Consider an eigenfunction in the form

$$f(x) = c\mathrm{e}^{ipx}, \tag{2.1.2}$$

in which p is a parameter characterizing the eigenfunction. The relation between λ and p is

$$\lambda = p. \tag{2.1.3}$$

Now we want to define $f(x)$ on a discrete domain $\{n\}$ of points equally spaced by a, so $n = x/a$. The continuous, Hermitian operator $-i\mathrm{d}/\mathrm{d}x$ must be replaced with a suitable discrete one: the operator we choose is

$$-i\frac{[f(n+1) - f(n-1)]}{2a}. \tag{2.1.4}$$

Equation (2.1.1) becomes

$$-\frac{i}{2}[f(n+1) - f(n-1)] = \tilde{\lambda} f(n), \tag{2.1.5}$$

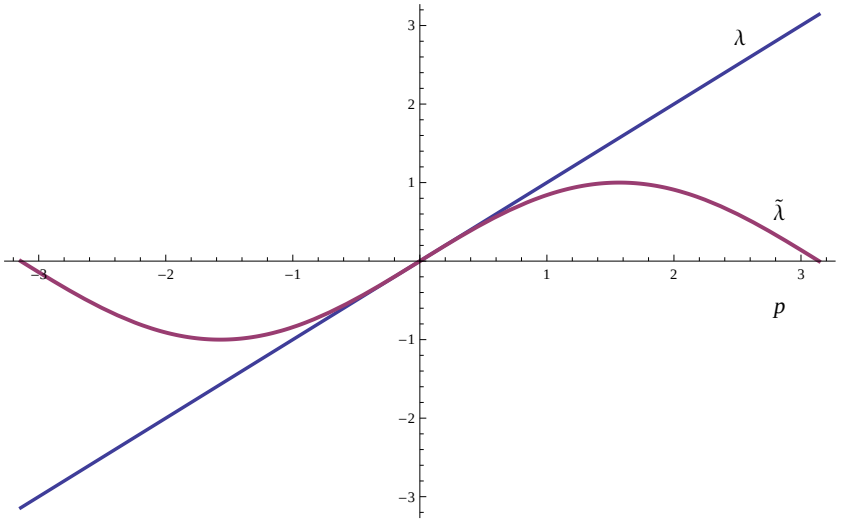

Figure 2.1: Relations between p and λ for the continuous case and $\tilde{\lambda}$ for the discrete case are shown. For the discrete case, we have put $a = 1$.

with $\tilde{\lambda} = a\lambda$. We are interested in an eigenfunction in a form equivalent to $f(x)$ defined in (2.1.6), namely $f(n)$ in the form

$$f(n) = c\mathrm{e}^{ipan}. \tag{2.1.6}$$

The relation between $\tilde{\lambda}$ and p is now

$$\tilde{\lambda} = \sin ap, \tag{2.1.7}$$

since by replacing it into (2.1.1) one gets

$$-\frac{i}{2}c\left[\mathrm{e}^{ipa(n+1)} - \mathrm{e}^{ipa(n-1)}\right] = c\mathrm{e}^{ipan}\,\sin ap. \tag{2.1.8}$$

In Figure 2.1 the two profiles of λ and $\tilde{\lambda}$ are shown as functions of p.

Note that while in the continuous case the spectrum is non degenerate, in the discrete case it is, since for $p \in [-\pi/a, \pi/a]$ there are two solutions to the equation $\sin ap = \tilde{\lambda}$, and therefore two eigenfuntions. In particular, for each value of $\tilde{\lambda}$, one value of p will be in the interval $[-\pi/2, \pi/2]$, while the other one will be outside this interval. As a limiting case, the values of momentum which correspond to $\tilde{\lambda} = 0$ are $p = 0$ and $p = \pm\pi$, which are the values at the corners of the interval. If

$$p = \frac{\arcsin\tilde{\lambda}}{a} \tag{2.1.9}$$

is the value of p in $[-\pi/2, \pi/s]$, the two eigenfunctions are:

$$f_\lambda^{(1)}(n) = A\mathrm{e}^{i\arcsin\tilde{\lambda}n} \tag{2.1.10}$$

$$f_\lambda^{(2)}(n) = B\mathrm{e}^{\pi - i\arcsin\tilde{\lambda}n} = B(-1)^n\mathrm{e}^{-i\arcsin\tilde{\lambda}n}. \tag{2.1.11}$$

The continuum limit yields the relation (2.1.3), since with fixed p one gets

$$\lim_{a\to 0}\frac{\tilde{\lambda}}{a}=\lim_{a\to 0}\frac{\sin ap}{a}=p. \tag{2.1.12}$$

This process applies to the lattice formulation of the Dirac field theory, and in particular to the Dirac equation [13]. Difficulties increase if we consider a four component Dirac field in the Minkowski space, since in a d-dimensional space one gets 2^d-1 additional solutions for a single physical solution of the continuous Dirac equation [7]. A formalization of this problem is given by a theorem from Nielsen and Nimonuya [12]: it states that the fermion doubling problem arises in the passage from a continuous to a discrete field theory, if one defines a Hermitian differential operator, preserving at the same time the locality and the translation invariance of the model.

2.1.2 Staggered fermions

Let us start by recalling the free Dirac Hamiltonian density in the classical field theory

$$\mathcal{H}_D=\bar{\Psi}(-i\gamma^i\partial_i+m)\Psi.$$

We consider henceforth the standard representation for the Dirac spinors and for the γ^μ matrices: it is obtained from the chiral representation by operating the unitary field transformation [9]

$$\Psi(x)\to\frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{1} & \mathbf{1}\\ -\mathbf{1} & \mathbf{1}\end{pmatrix}\Psi(x)\equiv U\Psi(x). \tag{2.1.13}$$

The γ^μ matrices in this representation are obtained by taking $U\gamma^\mu U^\dagger$ and are

$$\gamma^0=\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & -\mathbf{1}\end{pmatrix},\quad \gamma^i=\begin{pmatrix}\mathbf{0} & \sigma^i\\ -\sigma^i & \mathbf{0}\end{pmatrix}. \tag{2.1.14}$$

While the chiral representation is diagonal with respect to helicity eigenvectors, this representation is diagonal with respect to positive and negative energy solutions: the Dirac spinor is composed by a positive energy two-components spinor and by a negative energy spinor. We now define, in a $(1+1)$-dimensional space, a field which describes spinless fermions: we represent it with a two-components Dirac spinor [14],

$$\chi(x)=\begin{pmatrix}\chi^1(x)\\ \chi^2(x)\end{pmatrix}, \tag{2.1.15}$$

and the Dirac matrices for this model are

$$\gamma'^0\equiv\beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \gamma'^1=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{2.1.16}$$

The Dirac Hamiltonian operator is now

$$\mathcal{H} = -i\beta\gamma'^1 \frac{\partial}{\partial x^1} + m\beta \equiv -i\alpha^1 \frac{\partial}{\partial x^1} + m\beta, \tag{2.1.17}$$

with

$$\alpha^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.1.18}$$

It follows that the equations of motion for the two spinor components are

$$i\partial_t \chi^1(t, x^1) = -i\partial_{x^1} \chi^2(t, x^1) + m\chi^1(t, x^1), \tag{2.1.19}$$
$$i\partial_t \chi^2(t, x^1) = -i\partial_{x^1} \chi^1(t, x^1) - m\chi^2(t, x^1). \tag{2.1.20}$$

Let us now work in a discrete one-dimensional space, with spacing a between points; henceforth x will be the integer number labelling the site of the lattice, so the relation with the continuous coordinate is $x^1 = ax$. We keep, instead, the continuity of time. The equations of motion in the discrete space are

$$i\partial_t \chi^1_x = -\frac{i}{2a}(\chi^2_{x+1} - \chi^2_{x-1}) + m\chi^1_x \tag{2.1.21}$$
$$i\partial_t \chi^2_x = -\frac{i}{2a}(\chi^1_{x+1} - \chi^1_{x-1}) - m\chi^2_x. \tag{2.1.22}$$

We can observe two symmetric situations in the above equations: by supposing that x has even parity, in the first equation χ^1 is taken on even sites and χ^2 on odd sites, while in the second equation there is the opposite situation, with χ^2 taken on even sites and χ^1 on odd ones. From the preceding section we know that solving Eqs. (2.1.21) and (2.1.22) would yield a fermion doubling. We would obtain two solutions to the Dirac equation with same energy and two values of momentum p in $[-\pi/a, \pi/a]$, one in the interval $[-\pi/2a, \pi/2a]$, and the other one outside. To avoid the fermion doubling problem, we introduce the solution performed by Susskind. Let us start by giving up the locality in the definition of the fermion field, and by defining a new one-component field ξ_x by taking the first component of χ on even sites and the second component on odd sites, like in (2.1.21):

$$\xi_x = \begin{cases} \chi^1_x & \text{if } (-1)^x = 1 \\ \chi^2_x & \text{if } (-1)^x = -1. \end{cases} \tag{2.1.23}$$

This means that on even sites there are positive energy solutions, while negative energy ones correspond to odd sites. It follows that we are imposing a two-periodical superlattice structure on the original lattice, and therefore the momentum domain is reduced to $[-\pi/2a, \pi/2a]$, in which there is no degeneration in the spectrum. It follows from the definition of ξ_x that the equation of motion is

$$i\partial_t \xi_x = -\frac{i}{2a}(\xi_{x+1} - \xi_{x-1}) + m(-1)^x \xi_x, \tag{2.1.24}$$

and that the new Hamiltonian is [13]

$$\begin{aligned} H_{stagg} &= \sum_x \xi_x^\dagger \left[-\frac{i}{2a}(\xi_{x+1} - \xi_{x-1}) + m(-1)^x \xi_x \right] \\ &= -\frac{i}{2a} \sum_x \xi_x^\dagger \xi_{x+1} + \frac{i}{2a} \sum_x \xi_x^\dagger \xi_{x-1} + m \sum_x (-1)^x \xi_x^\dagger \xi_x \\ &= -\frac{i}{2a} \sum_x \xi_x^\dagger \xi_{x+1} + \frac{i}{2a} \sum_x \xi_{x+1}^\dagger \xi_x + m \sum_x (-1)^x \xi_x^\dagger \xi_x \\ &= -\frac{i}{2a} \sum_x \xi_x^\dagger \xi_{x+1} + \text{H.c.} + m \sum_x (-1)^x \xi_x^\dagger \xi_x. \end{aligned} \tag{2.1.25}$$

This approach has been used in [15] to compute the discrete Hamiltonian for a two-components spinor in a $SU(2)$ lattice gauge theory. As in the continuous case, H_{stagg} is symmetric under global $U(1)$ transformations of fields in the form

$$\xi_x \to \mathrm{e}^{i\alpha} \xi_x, \tag{2.1.26}$$

with α a real constant. When we quantize the theory, fields functions ξ_x becomes field operators which satisfy anticommutation relations:

$$\{\hat{\xi}_x, \hat{\xi}_{x'}\} = \{\hat{\xi}_x^\dagger, \hat{\xi}_{x'}^\dagger\} = 0, \qquad \{\hat{\xi}_x, \hat{\xi}_{x'}^\dagger\} = \delta_{x\,x'}. \tag{2.1.27}$$

The transformations (2.1.26) are implemented on lattice field operators by converting (1.2.11) and (1.2.12) in the discrete, one-dimensional space case, and the result is [7]

$$\hat{\xi}_x \to \prod_y \mathrm{e}^{-i\alpha \hat{\xi}_y^\dagger \hat{\xi}_y} \hat{\xi}_x \prod_z \mathrm{e}^{i\alpha \hat{\xi}_z^\dagger \hat{\xi}_z} = \mathrm{e}^{i\alpha} \hat{\xi}_x. \tag{2.1.28}$$

We finally observe that in the formalism of staggered fermions, particles trapped in even parity sites of the lattice have a positive mass energy, while particles in odd parity sites a negative one. The vacuum state relative to the free Dirac field is that with minimum energy, and corresponds to the state in which all odd parity sites are fulfilled, since they provide the negative energy states.

2.2 Minimal coupling and gauge transformations

2.2.1 Gauge field on the lattice

As in the continuum field model, the coupling of fermions field with a gauge field is introduced in order to promote the $U(1)$ symmetry of the Hamiltonian (2.1.25) from global to local. We will use the comparator defined in (1.1.48),

$$U(x, y) = \exp\left\{ i \int_x^y \mathrm{d}x^\mu A_\mu \right\},$$

in which A_μ is the four vector potential. The comparator transformation rule is given by (1.1.41):

$$U(x,y) \to \mathrm{e}^{i\alpha(x)} U(x,y) \mathrm{e}^{-i\alpha(y)}.$$

Now, let us recall the kinetic term in H_{stagg},

$$H_{kin} = -\frac{i}{2a} \sum_x \xi_x^\dagger \xi_{x+1} + \text{H.c.},$$

and observe that it transforms under a local $U(1)$ transformation as follows

$$H_{kin} \to H'_{kin} = -\frac{i}{2a} \sum_x \mathrm{e}^{-i\alpha_x} \mathrm{e}^{i\alpha_{x+1}} \xi_x^\dagger \xi_{x+1} + \text{H.c.}, \tag{2.2.1}$$

with α_x a real function. To implement the local $U(1)$ symmetry on H_{stagg} we have to define the comparator on the lattice links between the lattice sites. We consider the vector potential defined in the continuum space, in the gauge $A_0 = 0$; the only non vanishing component is $A^1(x^1) \equiv A(x^1)$. A general definition for the comparator on the lattice is

$$U(x, x+1) = \mathrm{e}^{-iaA(x^{1*})}, \tag{2.2.2}$$

in which a is the lattice spacing and x^{1*} is a suitable point in the interval $[xa, (x+1)a]$. Let us remark that in the literature there is an ambiguity about the choice of x^{1*}. In some texts [12, 14] the vector potential is considered as a function defined on the lattice, therefore the point x^{1*} can be only xa or $(x+1)a$. It is taken $x^{1*} \equiv xa$, namely x^{1*} coincides with the left corner of the interval. Other texts [16, 17] consider the vector potential defined in the continuous space, and adopt the so-called midpoint rule, according to which $x^{1*} \equiv xa + a/2$. This second formulation is adopted in the path integral quantization of nonrelativistic electrodynamics: it is the only choice with which the wave functions evolution obtained with the path integral formula is equivalent, in the continuum limit, with the evolution given by the Schrödinger equation. We have already dealt with this ambiguity in Chapter 1, when we passed from the first order expression for the comparator (1.1.46) to the second order one (1.1.53).

We adopt here the midpoint rule, therefore we define

$$U(x, x+1) = \mathrm{e}^{-iaA(xa+a/2)}. \tag{2.2.3}$$

Now, let us define a link variable for the vector potential which replaces the function $A(x^1)$: henceforth we set the lattice spacing $a = 1$ and define the variable $A_{x,x+1}$ by imposing [7]

$$U(x, x+1) = \mathrm{e}^{iA_{x,x+1}}, \tag{2.2.4}$$

and, given a real function α_x, it transforms according to the rule

$$U(x, x+1) \to \mathrm{e}^{i\alpha_x} U(x, x+1) \mathrm{e}^{-i\alpha_{x+1}}. \tag{2.2.5}$$

The gauge invariant expression for the kinetic term H_{kin} is therefore

$$H_{kinG} = -\frac{i}{2a}\sum_x \xi_x^\dagger U_{x,x+1}\xi_{x+1} + \text{H.c.}. \tag{2.2.6}$$

We now quantize the vector potential by promoting $A_{x,x+1}$ to a field operator, and define the electric field $\hat{E}_{x,x+1}$ as its conjugate variable: the algebra commutation rule they satisfy is the discrete version of (1.2.22), namely

$$[\hat{A}_{x,x+1}, \hat{E}_{x',x'+1}] = -i\delta_{x\,x'}. \tag{2.2.7}$$

Once we define the comparator in the quantized theory as $\tilde{U}_{x,x+1} = \mathrm{e}^{i\hat{A}_{x,x+1}}$, the commutation relation between the electric field and the comparator itself is

$$[\hat{E}_{x',x'+1}, \tilde{U}_{x,x+1}] = \delta_{x\,x'}\tilde{U}_{x,x+1}. \tag{2.2.8}$$

Let us redefine the comparator by including the i factor which appears in H_{kinG}; if we consider the quantized fermion field operators defined in (2.1.27) we can write the gauge invariant Hamiltonian density for the quantized theory,

$$\hat{H} = -\frac{1}{2a}\sum_x \hat{\xi}_x^\dagger \tilde{U}_{x,x+1}\hat{\xi}_{x+1} + \text{H.c.} + m\sum_x (-1)^x \hat{\xi}_x^\dagger \xi_x + \frac{g^2}{2}\sum_x \hat{E}_{x,x+1}^2. \tag{2.2.9}$$

The Hilbert space on which it acts is the tensor product of fermion number states on sites and electric field states on links; the first is finite dimensional, the second is infinite dimensional.

2.2.2 Gauge transformations

The generators of gauge transformations are the discrete version of the operators

$$\hat{\Psi}^\dagger\hat{\Psi}(x) - \nabla\cdot\hat{\mathbf{E}}(x)$$

in (1.2.33), with an additional term due to the use of staggered fermions. We define the operators on the lattice sites [7]

$$\hat{G}_x = \xi_x^\dagger\xi_x - (\hat{E}_{x,x+1} - \hat{E}_{x-1,x}) + \frac{1}{2}[(-1)^x - 1], \tag{2.2.10}$$

in which $(\hat{E}_{x,x+1} - \hat{E}_{x-1,x})$ is the discrete version of the electric field divergence. The local gauge transformations for the Hamiltonian are

$$\hat{H} \to \prod_x \mathrm{e}^{-i\alpha_x G_x}\hat{H}\prod_y \mathrm{e}^{i\alpha_y G_y}. \tag{2.2.11}$$

In the following we will prove the gauge invariance of the Hamiltonian by verifying that

$$[\hat{H}, \hat{G}_x] = 0 \qquad \forall x. \tag{2.2.12}$$

Let us first observe that the fermion operators and the gauge operators commute with each other, since they act on different spaces; also, the quadratic term in the electric field commute trivially with the generators, and we can ignore the phase factor due to staggered fermions. The commutators which must explicitly computed are

$$\left[\frac{1}{2a}\sum_x \hat{\xi}_x^\dagger \tilde{U}_{x,x+1}\hat{\xi}_{x+1} + \text{H.c.}, \hat{\xi}_y^\dagger \hat{\xi}_y\right], \tag{2.2.13}$$

$$\left[\frac{1}{2a}\sum_x \hat{\xi}_x^\dagger \tilde{U}_{x,x+1}\hat{\xi}_{x+1} + \text{H.c.}, (\hat{E}_{y,y+1} - \hat{E}_{y-1,y})\right], \tag{2.2.14}$$

$$\left[\sum_x \hat{\xi}_x^\dagger \hat{\xi}_x, \hat{\xi}_y^\dagger \hat{\xi}_y\right]. \tag{2.2.15}$$

The last commutator is zero, since for $x = y$ the two operators coincide and therefore commute; for $x \neq y$, instead,

$$\begin{aligned}
&[\hat{\xi}_x^\dagger \hat{\xi}_x, \hat{\xi}_y^\dagger \hat{\xi}_y] \\
&= \hat{\xi}_x^\dagger \hat{\xi}_x \hat{\xi}_y^\dagger \hat{\xi}_y - \hat{\xi}_y^\dagger \hat{\xi}_y \hat{\xi}_x^\dagger \hat{\xi}_x \\
&= -\hat{\xi}_x^\dagger \hat{\xi}_y \hat{\xi}_x^\dagger \hat{\xi}_y - \hat{\xi}_y^\dagger \hat{\xi}_y \hat{\xi}_x^\dagger \hat{\xi}_x = 0.
\end{aligned} \tag{2.2.16}$$

Commutator (2.2.13) yields

$$\begin{aligned}
&\left(\sum_x \hat{\xi}_x^\dagger \tilde{U}_{x,x+1}\hat{\xi}_{x+1} + \text{H.c.}\right)\hat{\xi}_y^\dagger \hat{\xi}_y - \hat{\xi}_y^\dagger \hat{\xi}_y \left(\sum_x \hat{\xi}_x^\dagger \tilde{U}_{x,x+1}\hat{\xi}_{x+1} + \text{H.c.}\right) \\
&= \sum_x \tilde{U}_{x,x+1}\hat{\xi}_x^\dagger(\delta_{x+1,y} - \hat{\xi}_y^\dagger \hat{\xi}_{x+1})\hat{\xi}_y + \sum_x \tilde{U}_{x,x+1}^\dagger \hat{\xi}_{x+1}^\dagger(\delta_{x,y} - \hat{\xi}_y^\dagger \hat{\xi}_x)\hat{\xi}_y \\
&\quad - \hat{\xi}_y^\dagger \hat{\xi}_y \left(\sum_x \hat{\xi}_x^\dagger \tilde{U}_{x,x+1}\hat{\xi}_{x+1} + \text{H.c.}\right) \\
&= \hat{\xi}_{y-1}^\dagger \tilde{U}_{y-1,y}\hat{\xi}_y + \hat{\xi}_{y+1}^\dagger \tilde{U}_{y,y+1}^\dagger \hat{\xi}_y - \hat{\xi}_y^\dagger \hat{\xi}_y \left(\sum_x \hat{\xi}_x^\dagger \tilde{U}_{x,x+1}\hat{\xi}_{x+1} + \text{H.c.}\right) \\
&\quad - \sum_x \left(\tilde{U}_{x,x+1}\hat{\xi}_x^\dagger \hat{\xi}_y^\dagger \hat{\xi}_{x+1}\hat{\xi}_y + \tilde{U}_{x,x+1}^\dagger \hat{\xi}_{x+1}^\dagger \hat{\xi}_y^\dagger \hat{\xi}_x \hat{\xi}_y\right) \\
&= \hat{\xi}_{y-1}^\dagger \tilde{U}_{y-1,y}\hat{\xi}_y + \hat{\xi}_{y+1}^\dagger \tilde{U}_{y,y+1}^\dagger \hat{\xi}_y - \hat{\xi}_y^\dagger \hat{\xi}_y \left(\sum_x \hat{\xi}_x^\dagger \tilde{U}_{x,x+1}\hat{\xi}_{x+1} + \text{H.c.}\right) \\
&\quad - \sum_x \left(\tilde{U}_{x,x+1}\hat{\xi}_y^\dagger(\delta_{x,\,y} - \hat{\xi}_y \hat{\xi}_x^\dagger)\hat{\xi}_{x+1} + \tilde{U}_{x,x+1}^\dagger \hat{\xi}_y^\dagger(\delta_{x+1,\,y} - \hat{\xi}_y \hat{\xi}_{x+1}^\dagger)\hat{\xi}_x\right) \\
&= \hat{\xi}_{y-1}^\dagger \tilde{U}_{y-1,y}\hat{\xi}_y + \hat{\xi}_{y+1}^\dagger \tilde{U}_{y,y+1}^\dagger \hat{\xi}_y - \hat{\xi}_y^\dagger \hat{\xi}_y \left(\sum_x \hat{\xi}_x^\dagger \tilde{U}_{x,x+1}\hat{\xi}_{x+1} + \text{H.c.}\right) \\
&\quad - \hat{\xi}_y^\dagger \tilde{U}_{y,y+1}\hat{\xi}_{y+1} - \hat{\xi}_y^\dagger \tilde{U}_{y-1,y}^\dagger \hat{\xi}_{y-1} + \sum_x \left(\tilde{U}_{x,x+1}\hat{\xi}_y^\dagger \hat{\xi}_y \hat{\xi}_x^\dagger \hat{\xi}_{x+1} + H.c\right) =
\end{aligned}$$

$$= \hat{\xi}^{\dagger}_{y-1} \tilde{U}_{y-1,y} \hat{\xi}_y + \hat{\xi}^{\dagger}_{y+1} \tilde{U}^{\dagger}_{y,y+1} \hat{\xi}_y - \hat{\xi}^{\dagger}_y \tilde{U}_{y,y+1} \hat{\xi}_{y+1} - \hat{\xi}^{\dagger}_y \tilde{U}^{\dagger}_{y-1,y} \hat{\xi}_{y-1}. \tag{2.2.17}$$

Commutator (2.2.14), instead, reads

$$\begin{aligned} &\sum_x \left[\hat{\xi}^{\dagger}_x [\tilde{U}_{x,x+1}, \hat{E}_{y,y+1}] \hat{\xi}_{x+1} - \hat{\xi}^{\dagger}_x [\tilde{U}_{x,x+1}, \hat{E}_{y-1,y}] \hat{\xi}_{x+1} \right] + [\mathrm{H.c.}, \hat{\mathrm{E}}_{\mathrm{y,y+1}}] \\ &= - \hat{\xi}^{\dagger}_y \tilde{U}_{y,y+1} \hat{\xi}_{y+1} + \hat{\xi}^{\dagger}_{y-1} \tilde{U}_{y-1,y} \hat{\xi}_y + \mathrm{H.c.}. \end{aligned} \tag{2.2.18}$$

From the definition of the Hamiltonian (2.2.9) it follows that the commutator (2.2.12) is obtained by subtracting between each other the results in (2.2.17) and (2.2.18); since they coincide, the commutator (2.2.12) vanishes.

In conclusion, we have defined the Hamiltonian for a quantized Abelian gauge theory on a one dimensional lattice, in which the gauge field is coupled with a two components Dirac field. We will use this model in the next chapter, since this is the system whose evolution we want simulate.

Chapter 3

A Quantum simulator with ultracold atoms in an optical lattice

In this chapter we will explain how it is possible to realize a quantum simulator for an Abelian gauge theory with a cloud of ultracold atoms trapped in an optical lattice. First, the Quantum Link Model (QLM) will be introduced, in which an electric field operator with a finite spectrum is defined. Then, it will be shown how the Hamiltonian of an Abelian gauge theory can emerge as an effective Hamiltonian, up to second order perturbations, from that of an ultracold atoms system in an optical lattice; to do this, we will define the degrees of freedom of the Abelian theory in the physical system of the simulator. This chapter follows the same logical steps of the article by Banerjee *et al.* [2].

3.1 The Quantum Link Model

The physical system we are dealing with is an optical lattice in which ultracold atoms are trapped. Due to tunnelling, the atoms hop from a site of the lattice to an adjacent one with an amplitude fixed by the tunnelling amplitude. Many species of atoms can be used, and different atoms can interact in different ways with the optical lattice, and therefore hop with different amplitudes between lattice sites.

The dynamics we want to implement with this simulator is described by the following Hamiltonian [2],

$$H = -t \sum_x \psi_x^\dagger \tilde{U}_{x,x+1} \psi_{x+1} + \text{H.c.} + m \sum_x (-1)^x \psi_x^\dagger \psi_x + \frac{g^2}{2} \sum_x \hat{E}_{x,x+1}^2, \quad (3.1.1)$$

in which x labels lattice sites. Let us recall that the operators ψ_x are fermion field operators, obeying anticommutation relations

$$\{\psi_x, \psi_{x'}^\dagger\} = \delta_{x,x'}, \quad \{\psi_x, \psi_{x'}\} = 0, \quad \{\psi_x^\dagger, \psi_{x'}^\dagger\} = 0. \quad (3.1.2)$$

Operator $\tilde{U}_{x,x+1}$ is the unitary comparator defined in Chapter 2, $\hat{E}_{x,x+1}$ is the electric field operator, and the commutation rule they satisfy is

$$[\hat{E}_{x,x+1}, \tilde{U}_{x',x'+1}] = \delta_{x\,x'}\tilde{U}_{x,x+1}. \tag{3.1.3}$$

In Equation (3.1.1), t is the transition amplitude for fermion hopping, m the fermion mass and g a coupling constant for the electric field. Since we are studying a one dimensional system we do not consider a magnetic field energy term. The phase factor $(-1)^x$ in the mass term is due to the use of staggered fermions.

The Hilbert space on which the Hamiltonian acts is the tensor product of the Hilbert spaces relative to each site and each link. In particular, let us observe that the spaces relative to links are infinite-dimensional, since they contain the eigenvectors of the electric field, which form an infinite orthonormal basis.

The simulation of this theory with ultracold atoms first requires the definition of some variables of the microscopic system which represent the matter and radiation fields of the gauge theory. This variables admit a finite and discrete set of values, therefore also the electric field will be represented by a quantity which can take only a finite set of values. This means that the model, and the Hamiltonian (3.1.1), must be modified *ad hoc*.

The solution we adopt is that of the Quantum Link Model [7]. In this model, one replaces the comparator and the electric field operators with two finite and discrete operators which satisfy the same commutation relation (3.1.3): we fix the dimension n of the link Hilbert spaces and consider n-dimensional spin operators S^i, with $i = 1, 2, 3$. If $\sqrt{S(S+1)}$ is the modulus of the spin we have $n = 2S+1$. The raising operator for the eigenvectors of the third spin component S^3 is $S^+ = S^1 + iS^2$, and obeys the commutation relation

$$[S^3, S^+] = S^+. \tag{3.1.4}$$

By defining on each link the following operators,

$$U_{QLM\,x,x+1} = S^+_{x,x+1}, \tag{3.1.5}$$

$$\hat{E}_{QLM\,x,x+1} = S^3_{x,x+1}; \tag{3.1.6}$$

the electric field is represented by a discrete operator whose spectrum is

$$\{-S, \ldots, +S\}, \tag{3.1.7}$$

and the commutation relation (3.1.3) is still valid, since

$$[\hat{E}_{QLM\,x,x+1}, U_{QLM\,x',x'+1}] = [S^3_{x,x+1}, S^+_{x',x'+1}] = \delta_{x\,x'}U_{QLM\,x,x+1}. \tag{3.1.8}$$

Henceforth the Gauge theory Hamiltonian we will refer to is

$$H_{QLM} = \\ -t\sum_x \psi^\dagger_x U_{QLM\,x,x+1}\psi_{x+1} + \text{h.c.} + m\sum_x (-1)^x \psi^\dagger_x \psi_x + \frac{g^2}{2}\sum_x \hat{E}^2_{QLM\,x,x+1}. \tag{3.1.9}$$

Let us define the generators of gauge transformations in the Quantum Link Model, which are

$$G_x = \psi_x^\dagger \psi_x - (\hat{E}_{QLM\,x,x+1} - \hat{E}_{QLM\,x-1,x}) + \frac{1}{2}[(-1)^x - 1]. \tag{3.1.10}$$

The system is $U(1)$ gauge invariant, since $[G_x, H_{QLM}] = 0$: the demonstration is similar to that performed in Chapter 2. The validity of the commutation rule (3.1.3) in the passage from the continuous field model to the QLM is important, since it guarantees the $U(1)$ gauge invariance of the theory.

The gauge invariance is also the criterion with which physical states are selected: as we have seen in Chapter 1, in Equation (1.2.33), physical states are represented by vectors which are annihilated by all the generators. Also in the QLM we define the physical states as those vectors $|\Phi\rangle$ satisfying

$$G_x|\Phi\rangle = 0 \qquad \forall x. \tag{3.1.11}$$

This condition leads to a local equation for the eigenvalues of the operators: for each site we have that [2]

$$n_x^F - (E_{QLM\,x,x+1} - E_{QLM\,x-1,x}) + \frac{1}{2}[(-1)^x - 1] = 0, \tag{3.1.12}$$

with n_x^F the fermion number eigenvalue. Since henceforth we will refer only to the operators of the QLM, the relative labels will be omitted.

3.2 The implementation of gauge theory degrees of freedom

In this section we will show how to implement system the degrees of freedom of the Abelian gauge theory on the lattice. The degrees of freedom are the fermion matter field, the comparator and the electric field.

We need three species of atoms on the lattice. A fermionic species will be used to define the matter field, while the other two are necessary to implement the electric field and the comparator. So let us define the field operators relative to these atomic species. For the fermionic species we introduce the operator ψ_x, which satisfies the anticommutation rules (3.1.2). These operators describe an atom species which behaves as a fermion on the lattice. We use the same symbol of the fermion fields in the gauge theory Hamiltonian since they can be identified. The field operators relative to the two other species satisfy commutation relation, therefore the atoms behave like bosons on the lattice. The operators are b_x^σ, with $\sigma = 1, 2$ distinguishing the two species [2]. The commutation relations are

$$[b_x^\sigma, b_{x'}^{\sigma'}] = 0, \quad [b_x^{\sigma\dagger}, b_{x'}^{\sigma'\dagger}] = 0, \quad [b_x^\sigma, b_{x'}^{\sigma'\dagger}] = \delta_{\sigma\,\sigma'}\delta_{x\,x'}. \tag{3.2.1}$$

The number operators for each species are n_x^α, with $\alpha = 1, 2, F$ with

$$n_x^1 = b_x^{1\dagger} b_x^1, \quad n_x^2 = b_x^{2\dagger} b_x^2, \quad n_x^F = \psi_x^\dagger \psi_x. \tag{3.2.2}$$

The microscopic Hilbert space $\mathcal{H}$ is the tensor product of the Hilbert spaces associated atoms of each species on each site: the basis we choose is

$$\left\{ \prod_x |n_x^F;\, n_x^1;\, n_x^2\rangle \right\}, \tag{3.2.3}$$

which is obtained from the tensor product of the number operators eigenvectors.

The Hubbard microscopic Hamiltonian which rules the dynamic of the atoms on the lattice reads:

$$\begin{aligned} H_{micro} = & - t_B \sum_{x \text{ even}} b_{x+1}^{1\dagger} b_x^1 - t_B \sum_{x \text{ odd}} b_{x+1}^{2\dagger} b_x^2 - t_F \sum_x \psi_{x+1}^\dagger \psi_x + \text{H.c.} \\ & + \sum_{x,\alpha,\beta} n_x^\alpha U_{\alpha\,\beta} n_x^\beta + \sum_{x,\alpha} (-1)^x U_\alpha n_x^\alpha. \end{aligned} \tag{3.2.4}$$

The first three terms, and their Hermitian conjugates, are the hopping terms: fermions can hop from a site to the adjacent one with an amplitude t_F. Bosons of species 1 can hop only from a site with even parity to the adjacent site on the right, and viceversa from a site with odd parity to the adjacent site on the left. Conversely, bosons of species 2 hop from a site with odd parity to the adjacent right one, and from a site with even parity to the adjacent left site.

The optical lattice potential is shaped as a superlattice to allow only this transitions; both species of bosons have the same transition amplitude t_B.

Summarizing, each fermion is free to move along the whole lattice, while each boson is confined on a fixed couple of sites, which can be identified with a link: for example, a boson of species 1 initially in a even site can hop toward right, and then its only allowed transition is coming back to the original position.

The term $\sum_{x,\alpha,\beta} n_x^\alpha U_{\alpha\,\beta} n_x^\beta$ is an on site repulsion between atoms of different species, with $U_{\alpha\,\beta}$ coupling constants, and $\sum_{x,\alpha} (-1)^x U_\alpha n_x^\alpha$ fixes a difference in potential energy between atoms of the same species in adjacent sites. In Figure 3.1 this situation is shown, since the potentials for the two species are displayed: each bosonic atom can tunnel only between the two sites of the link in which is placed. In Figure 3.2 the potential profile of fermions is displayed: they are allowed to tunnel with the amplitude t_F along the whole lattice, as long as the exclusion principle is satisfied.

Now we can define the electric field and the comparator on the lattice by using the formalism of Schwinger's oscillator model of angular momentum [18]. Let us consider for definiteness a link, $x, x+1$, with an even parity site on the left: bosons of species 1 can tunnel between the sites of this link. We assign $2S$ bosons of species 1 to this link: these atoms are free to stay on one site, or on

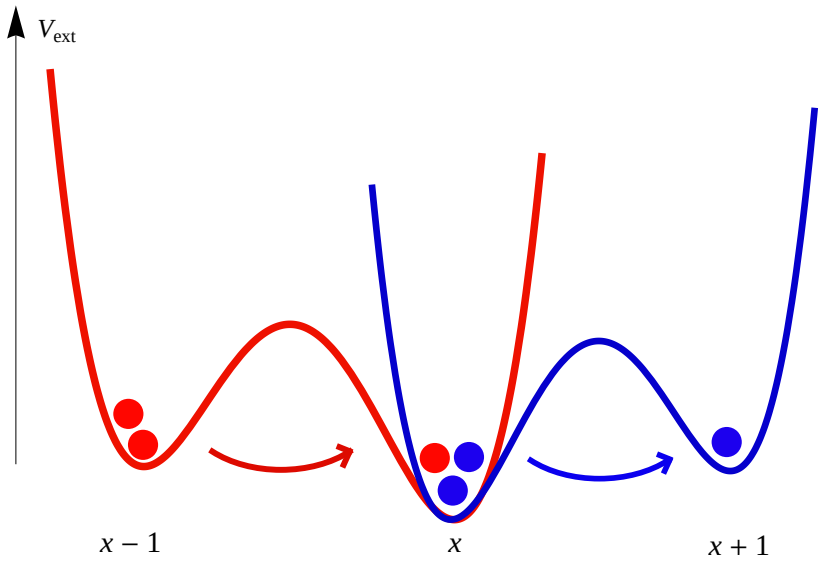

Figure 3.1: The external potential V_{ext} of bosons in the lattice are displayed; the site x has odd parity, and $S = 3/2$. Red dots are bosons of species 1, blue dots of species 2. The difference between minima of adjacent sites recalls the energetic offset.

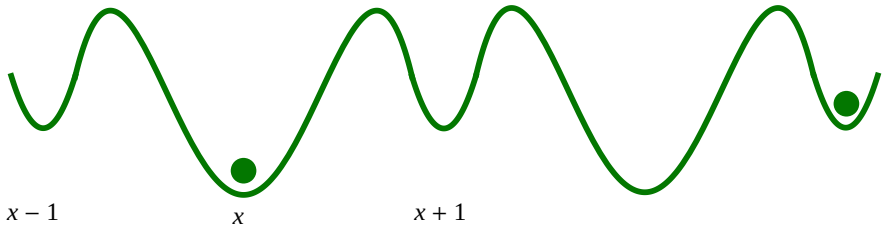

Figure 3.2: The potential profile of fermions is displayed. Site x has odd parity, and the difference between minima of adjacent sites corresponds to the energy offset.

the other. We define the following electric field operator on each link with an even left site parity:

$$E^1_{x,x+1} = \frac{1}{2}\left[b^{1\dagger}_{x+1}b^1_{x+1} - b^{1\dagger}_x b^1_x\right]. \tag{3.2.5}$$

Its eigenvalues are the same of the QLM electric field, $\{-S,\dots,+S\}$. This procedure can be repeated for links with odd-parity sites on the left: in this case we confine $2S$ atoms of species 2 on the two sites of the links and define electric field operator in the same way. A general definition therefore can be given:

$$E^\sigma_{x,x+1} = \frac{1}{2}\left[b^{\sigma\dagger}_{x+1}b^\sigma_{x+1} - b^{\sigma\dagger}_x b^\sigma_x\right], \quad \sigma = \begin{cases} 1 & \text{if } (-1)^x = 1 \\ \\ 2 & \text{if } (-1)^x = -1. \end{cases} \tag{3.2.6}$$

In Figure 3.1 there are $2S$ atoms for each species placed on the relative links; the electric field is $E_{x-1,x} = -1/2$ on the first link, and again $E_{x,x+1} = -1/2$ on the second. Finally, the arrows indicate the direction of tunnelling which increases the electric field on the two links. From the definition of the electric field it follows that the comparator acts, in this implementation, by moving a boson from the site x to the site $x+1$: we define the operator

$$U^\sigma_{x,x+1} = b^{\sigma\dagger}_{x+1}b^\sigma_x, \quad \sigma = \begin{cases} 1 & \text{if } (-1)^x = 1 \\ \\ 2 & \text{if } (-1)^x = -1. \end{cases} \tag{3.2.7}$$

Let us show the form of Gauss' law in terms of the atomic field operators with the definition (3.2.6) of the electric field: for $(-1)^x = 1$, the expression for the generators reads

$$\begin{aligned} G^\sigma_x =& n^F_x - (E^\sigma_{x,x+1} - E^{\sigma'}_{x-1,1}) + \frac{1}{2}[(-1)^x - 1] \\ =& n^F_x - (E^1_{x,x+1} - E^2_{x-1,1}) \\ =& n^F_x - \frac{1}{2}[n^1_{x+1} - n^1_x - (n^2_x - n^2_{x-1})] \\ =& n^F_x - \frac{1}{2}[(2S - n^1_x) - n^1_{x+} - (n^2_x - (2S - n^2_x))] \\ =& n^F_x - \frac{1}{2}[2S - 2n^1_x + 2S - 2n^2_x] \\ =& n^F_x + n^1_x + n^2_x - 2S, \end{aligned} \tag{3.2.8}$$

therefore the condition on the eigenvalues is

$$n^F_x + n^1_x + n^2_x - 2S = 0, \qquad \forall x. \tag{3.2.9}$$

The gauge invariance condition fixes the total number of atoms which can be placed on each link. The calculation performed for $(-1)^x = 1$ can be extended to the general case, yielding the condition

$$n_x^F + n_x^1 + n_x^2 - 2S + \frac{1}{2}[(-1)^x - 1] = 0. \tag{3.2.10}$$

3.3 Implementation of the dynamics

3.3.1 Analysis of the microscopic Hamiltonian

Let us recall the microscopic Hamiltonian (3.2.4):

$$\begin{aligned} H_{micro} = & - t_B \sum_{x \text{ even}} b_{x+1}^{1\dagger} b_x^1 - t_B \sum_{x \text{ odd}} b_{x+1}^{2\dagger} b_x^2 - t_F \sum_x \psi_{x+1}^\dagger \psi_x + \text{H.c.} \\ & + \sum_{x,\alpha,\beta} n_x^\alpha U_{\alpha\,\beta} n_x^\beta + \sum_{x,\alpha} (-1)^x U_\alpha n_x^\alpha. \end{aligned}$$

The atoms trapped in the optical lattice evolve according to this Hamiltonian. After we have defined on the lattice the electric field and the fermion field in terms of atomic field operators, we can construct an initial state obeying condition (3.2.10) and let it evolve. There is a finite probability that during its evolution the state looses its gauge invariance: this means that we are not exactly simulating the evolution of a gauge theory with a coupling to fermions. In this section we will show that with a proper definition of the parameters in the Hamiltonian (3.2.4) the evolution of a gauge invariant state of the microscopic system approximates the one expected in a gauge theory evolving with the QLM Hamiltonian (3.1.9). In the following we will make use of this Hamiltonian:

$$\begin{aligned} \tilde{H} = & H + U \sum_x G_x^2 \\ = & - t \sum_x \psi_x^\dagger \tilde{U}_{x,x+1} \psi_{x+1} + \text{H.c.} + m \sum_x (-1)^x \psi_x^\dagger \psi_x \\ & + \frac{g^2}{2} \sum_x \hat{E}_{x,x+1}^2 + U \sum_x G_x^2, \end{aligned} \tag{3.3.1}$$

with U a positive constant larger than the other parameters in H, namely

$$U \gg m, t, g. \tag{3.3.2}$$

The Hamiltonian $\tilde{H}$ differs from H for the term $U \sum_x G_x^2$: given a physical, gauge invariant state $|\Phi\rangle$ there is no difference in the action of these two operators,

$$\tilde{H}|\Phi\rangle = H|\Phi\rangle, \tag{3.3.3}$$

since $G_x|\Phi\rangle = 0$.

By considering instead a non gauge invariant vector $|\Psi\rangle$, one gets

$$U \sum_x G_x^2 |\Psi\rangle \neq 0.$$

Due to condition (3.3.2) the energy corresponding to this vector is of a larger order of magnitude than that of gauge invariant states, whose energies are of the order of m, t, g: it follows that non-gauge invariant states are energetically more difficult to realize. If we consider the gauge theory at low energies, the term $U \sum_x G_x^2$ works as a constraint which keeps the evolution of the system in the gauge invariant subspace of the total Hilbert space.

Even if we are focusing on the use of a quadratic term in the generators to implement a constraint on the evolution, this is not the only possibility to control a gauge invariant evolution of the system: for example, it has been studied how a white noise term could be used as well [3].

To apply second order perturbation theory, let us distinguish in the Hamiltonian (3.2.4) two parts, one which preserves gauge invariance of states, and one which breaks it. Therefore we write

$$H_{micro} = H_U + \Delta H, \tag{3.3.4}$$

with [2]

$$\begin{aligned} H_U =& (U + \frac{g^2}{4}) \sum_{x,\sigma} (n_x^\sigma)^2 + 2U \sum_x n_x^1 n_x^2 + U \sum_{x,\sigma} (-1)^x n_x^\sigma \\ &+ 2U \sum_{x,\sigma} n_x^F n_x^\sigma + (U + m) \sum_x (-1)^x n_x^F, \end{aligned} \tag{3.3.5}$$

$$\begin{aligned} \Delta H =& - t_F \sum_x (\psi_{x+1}^\dagger \psi_x + \psi_x^\dagger \psi_{x+1}) - t_B \sum_{x \text{ odd}} (b_{x+1}^{2\dagger} b_x^2 + b_x^{2\dagger} b_{x+1}^2) \\ &- t_B \sum_{x \text{ even}} (b_{x+1}^{1\dagger} b_x^1 + b_x^{1\dagger} b_{x+1}^1). \end{aligned} \tag{3.3.6}$$

The term H_U is obtained from the three last terms of H_{micro}, with a suitable tuning of the parameters $U_{\alpha\beta}$ and U_α; this term is a gauge invariant term, since it contains number operators and does not modify the number of particle on each site. If $|\Phi\rangle$ is a gauge invariant state, $H_U|\Phi\rangle$ is still gauge invariant, and by the other side if $|\Phi'\rangle$ is not gauge invariant, $H_U|\Phi'\rangle$ is not gauge invariant either.

The term ΔH is given by the first three terms of (3.2.4) with their Hermitian conjugates, and it is responsible of transitions from gauge invariant to non gauge invariant states and viceversa: by applying one of the hopping terms to a gauge invariant state $|\Phi\rangle$, the local condition $G_x|\Phi\rangle = 0$ will no longer be valid.

Now we will apply second order perturbation theory to make the Hamiltonian H emerge from H_{micro}. The first step is to demonstrate that, up to a constant

factor, we can write H_U defined in (3.3.8) in the form

$$H_U = \frac{g^2}{2}\sum_x (E_{x,x+1})^2 + m\sum_x (-1)^x \psi_x^\dagger \psi_x + U\sum_x (G_x)^2, \tag{3.3.7}$$

in which we have synthetically omitted the upper indices σ on $E_{x,x+1}$ and G_x relative to boson species and to the parity of x. First, we expand and rearrange H_U:

$$\begin{aligned} H_U &= \left(U + \frac{g^2}{4}\right)\sum_x \left[(n_x^1)^2 + (n_x^2)^2\right] + 2U\sum_x n_x^1 n_x^2 + U\sum_x (-1)^x \left[n_x^1 + n_x^2\right] \\ &\quad + 2U\sum_x \left(n_x^1 n_x^F + n_x^2 n_x^F\right) + (U+m)\sum_x (-1)^x n_x^F \\ &= \frac{g^2}{2}\sum_x \frac{1}{2}\left[(n_x^1)^2 + (n_x^2)^2\right] + m\sum_x (-1)^x n_x^F + U\sum_x [(n_x^1)^2 + (n_x^2)^2 \\ &\quad + 2\left(n_x^1 n_x^2 + n_x^1 n_x^F + n_x^F n_x^2\right) + (-1)^x (n_x^F + n_x^1 + n_x^2)]. \end{aligned} \tag{3.3.8}$$

Now, let us consider the quadratic term in the generators of $\tilde{H}$: by recalling the expression (3.2.8) one gets

$$\begin{aligned} \sum_x (G_x)^2 &= \sum_x \left(n_x^1 + n_x^2 + n_x^F - 2S + \frac{1}{2}[(-1)^x - 1]\right)^2 \\ &= \sum_x [(n_x^1)^2 + (n_x^2)^2 + \underbrace{(n_x^F)^2}_{(a)} + \underbrace{4S^2}_{b} + \frac{1}{2}[(-1)^x - 1] \\ &\quad + 2(n_x^1 n_x^2 + n_x^F n_x^2 + n_x^1 n_x^F) + (-1)^x (n_x^1 + n_x^2 + n_x^F) \\ &\quad - \underbrace{4S(n_x^1 + n_x^2 + n_x^F)}_{b} - (\underbrace{n_x^1 + n_x^2}_{b} + \underbrace{n_x^F}_{(a)}) - \underbrace{2S[(-1)^x - 1]}_{b}]. \end{aligned} \tag{3.3.9}$$

Some terms in the above expression vanish: by observing that $(n_x^F)^2 = n_x^F$, operators (a) in (3.3.9) cancels out. Also, the sum of terms (b) is constant, since the total number atoms of the two species on the lattice is fixed, and therefore we can ignore them. In conclusion, up to constant factors, we obtain

$$\sum_x (G_x)^2 = \sum_x [(n_x^1)^2 + (n_x^2)^2] + 2(n_x^1 n_x^2 + n_x^F n_x^2 + n_x^1 n_x^F) + (-1)^x (n_x^1 + n_x^2 + n_x^F)]. \tag{3.3.10}$$

Let us finally examine the electric field energy term: by applying the definition (3.2.6) it follows that

$$\begin{aligned} \sum_x (E_{x,x+1})^2 &= \sum_{x \text{ even}} (E_{x,x+1}^1)^2 + \sum_{x \text{ odd}} (E_{x,x+1}^2)^2 \\ &= \sum_{x \text{ even}} \frac{1}{4}(n_{x+1}^1 - n_x^1)^2 + \sum_{x \text{ odd}} \frac{1}{4}(n_{x+1}^2 - n_x^2)^2 = \end{aligned}$$

$$
\begin{aligned}
&= \sum_{x \text{ even}} \frac{1}{4}[(n^1_{x+1}) - 2n^1_{x+1}n^1_x + (n^1_x)^2] \\
&\quad + \sum_{x \text{ odd}} \frac{1}{4}[(n^2_{x+1})^2 - 2n^2_{x+1}n^2_x + (n^2_x)^2] \\
&= \sum_{x \text{ even}} \frac{1}{4}[(n^1_{x+1})^2 + (n^1_x)^2 + (n^1_{x+1})^2 - 4S^2 + (n^1_x)^2] \\
&\quad + \sum_{x \text{ odd}} \frac{1}{4}[(n^2_{x+1})^2 + (n^2_x)^2 + (n^2_{x+1})^2 - 4S^2 + (n^2_x)^2] \\
&= \sum_{x \text{ even}} \frac{1}{2}[(n^1_{x+1})^2 + (n^1_x)^2 - 2S^2] + \sum_{x \text{ odd}} \frac{1}{2}[(n^2_{x+1})^2 + (n^2_x)^2 - 2S^2] \\
&= \sum_x \frac{1}{2}[(n^1_x)^2 + (n^2_x)^2 - 2S^2], \qquad (3.3.11)
\end{aligned}
$$

in which we have used the condition

$$
n^\sigma_x + n^\sigma_{x+1} = 2S \Rightarrow -2n^\sigma_x n^\sigma_{x+1} = (n^\sigma_x)^2 + (n^\sigma_{x+1})^2 - 4S^2. \qquad (3.3.12)
$$

By putting together the results obtained in (3.3.11) and (3.3.10), and comparing them with (3.3.8), we obtain, up to a constant factor, the equality (3.3.7).

3.3.2 Setting up the second order perturbation theory

Let us observe that in section (3.3.1) we have found all terms of $\tilde{H}$ in H_U, except the kinetic terms of fermion matter fields,

$$
-t \sum_x \psi^\dagger_x \tilde{U}_{x,x+1} \psi_{x+1} + \text{H.c.}.
$$

We want to examine the gauge invariant structure of these operators in terms of the boson field operators: from the definition (3.2.7) of the comparator it follows that

$$
\begin{aligned}
&- t \sum_x \psi^\dagger_x U_{x,x+1} \psi_{x+1} + \text{H.c.} \\
&= - t \sum_{x \text{ even}} \psi^\dagger_x U^1_{x,x+1} \psi_{x+1} - t \sum_{x \text{ odd}} \psi^\dagger_x U^2_{x,x+1} \psi_{x+1} + \text{H.c.} \\
&= - t \sum_{x \text{ even}} \psi^\dagger_x b^{1\dagger}_{x+1} b^1_x \psi_{x+1} - t \sum_{x \text{ odd}} \psi^\dagger_x b^{2\dagger}_{x+1} b^2_x \psi_{x+1} + \text{H.c.}. \qquad (3.3.13)
\end{aligned}
$$

Each term is composed by four operators which act on a link; for each site of the link there are two operators, a creation and an annihilation one. This structure ensures that the gauge invariance of states is preserved, since the hopping of different atoms does not alter their total number on each site.

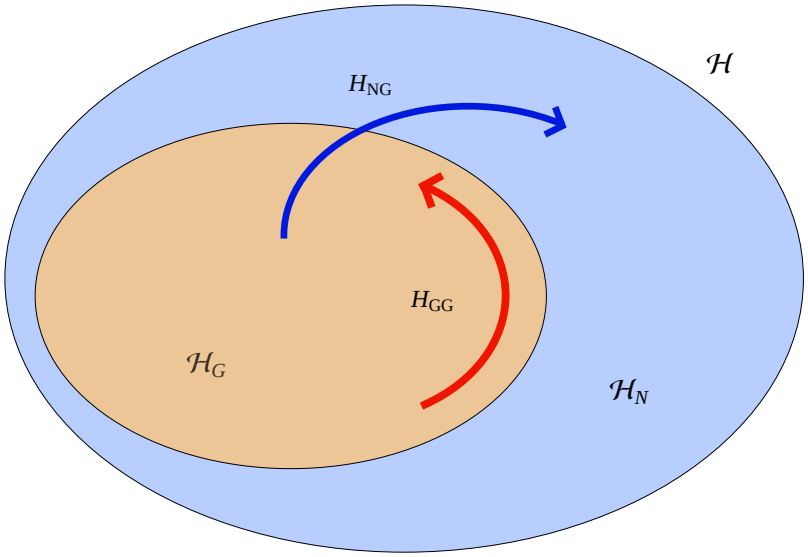

Figure 3.3: The total Hilbert space is divided into the two subspaces $\mathcal{H}_G$ and $\mathcal{H}_N$. The red arrow shows the gauge invariant evolution, the blue arrows reprsents the gauge breaking one.

Now we will obtain the Hamiltonian H as the effective Hamiltonian for a gauge invariant state by applying second order perturbation theory to the Hamiltonian H_{micro}; we will consider terms up to the second order with respect to the parameters m, t_F, t_B. The method we will use is that provided by the projection operators formalism [19].

The first step is to define two orthogonal projection operators acting on the total microscopic Hilbert space $\mathcal{H}$: $\hat{G}$ projects operator on gauge invariant states, while its complementary projection operator is

$$\hat{N} = \mathbb{1} - \hat{G}. \tag{3.3.14}$$

The Hilbert space $\mathcal{H}$ is decomposed into two subspaces, $\mathcal{H}_G$ and $\mathcal{H}_N$. A vector $|\Psi\rangle$ in $\mathcal{H}$ can be written as follows:

$$|\Psi\rangle = |\Psi_G\rangle + |\Psi_N\rangle = \begin{pmatrix} |\Psi_G\rangle \\ |\Psi_N\rangle \end{pmatrix}. \tag{3.3.15}$$

Since the basis vectors (3.2.3) are eigenvectors of the number operators, the basis itself is divided into two sets, one spanning $\mathcal{H}_G$ and the other spanning $\mathcal{H}_N$.

The Hamiltonian H_{micro} can also be decomposed according to these projection operator:

$$H_{micro} = \begin{pmatrix} \hat{G}H_{micro}\hat{G} & \hat{G}H_{micro}\hat{N} \\ \hat{N}H_{micro}\hat{G} & \hat{N}H_{micro}\hat{N} \end{pmatrix} = \begin{pmatrix} H_{GG} & H_{GN} \\ H_{NG} & H_{NN} \end{pmatrix}. \tag{3.3.16}$$

In Figure 3.3 the evolution of the system is displayed, in relation to the gauge invariant and non gauge invariant subspaces of $\mathcal{H}$. Let us consider an eigenvector

$|\Phi\rangle$ of the Hamiltonian H_{micro}; we suppose its energy E is obtained by perturbing the spectrum of the gauge invariant Hamiltonian.

The eigenvalue equation is

$$E \begin{pmatrix} |\Psi_G\rangle \\ |\Psi_N\rangle \end{pmatrix} = \begin{pmatrix} H_{GG} & H_{GN} \\ H_{NG} & H_{NN} \end{pmatrix} \begin{pmatrix} |\Psi_G\rangle \\ |\Psi_N\rangle \end{pmatrix}, \tag{3.3.17}$$

from which we put

$$E|\Psi_G\rangle = H_{GG}|\Psi_G\rangle + H_{GN}|\Psi_N\rangle, \tag{3.3.18}$$
$$E|\Psi_N\rangle = H_{NG}|\Psi_G\rangle + H_{NN}|\Psi_N\rangle. \tag{3.3.19}$$

From the second equation one obtains

$$|\Psi_N\rangle = \frac{1}{E - H_{NN}} H_{NG}|\Psi_G\rangle \simeq (H_{NN})^{-1} H_{NG}|\Psi_G\rangle, \tag{3.3.20}$$

in which the approximation is justified by observing that $H_{NG}|\Psi_G\rangle$ is a not gauge invariant state, and therefore the correspondent eigenvalues of H_{NN} have a larger order of magnitude than those of the gauge invariant Hamiltonian: let us recall indeed that the eigenvalues of non gauge invariant states are of order U, greater than all the other parameters in H_{micro}. The final equation of $|\Psi_G\rangle$ is obtained by substituting the above result in (3.3.18):

$$E|\Psi_G\rangle \simeq \left[H_{GG} + H_{GN}(H_{NN})^{-1} H_{NG}\right] |\Psi_G\rangle \equiv H_{eff}|\Psi_G\rangle. \tag{3.3.21}$$

In the next paragraph we will compute H_{eff}, showing how $\tilde{H}$ emerges from it.

3.3.3 The effective Hamiltonian

Let us begin from the calculation of the components (3.3.16) of the Hamiltonian H_{micro}. From (3.3.4) and (3.3.7) we have

$$H_{GG} = \hat{G} H_{micro} \hat{G} = \hat{G} \left[\frac{g^2}{2} \sum_x (E_{x,x+1})^2 + m \sum_x (-1)^x n_x^F \right] \hat{G}, \tag{3.3.22}$$

since

$$G_x \hat{G} = 0, \qquad \hat{G} \Delta H \hat{G} = 0.$$

The off-diagonal components of the Hamiltonian are

$$H_{GN} = \hat{G} \Delta H \hat{N}, \tag{3.3.23}$$
$$H_{NG} = \hat{N} \Delta H \hat{G}, \tag{3.3.24}$$

and they are the terms of the Hamiltonian which transforms the basis vectors (3.2.3): all the others terms of H_{micro} are diagonal in the chosen basis. Finally, there are no vanishing term of H_{micro} when we consider

$$H_{NN} = \hat{N} H_{micro} \hat{N}, \tag{3.3.25}$$

but we assume the approximation

$$H_{NN} \simeq -U\hat{N}\sum_x (G_x)^2\hat{N}, \tag{3.3.26}$$

namely, we omit the diagonal terms of lower order of magnitude, and ignore transitions between non gauge invariant states. So far, the effective Hamiltonian reads

$$\begin{aligned} H_{eff} =& \hat{G}\left[\frac{g}{2}\sum_x (E_{x,x+1})^2 + m\sum_x (-1)^x n_x^F\right]\hat{G} \\ &- \hat{G}\Delta H\hat{N}\frac{1}{U\sum_x (G_x)^2}\hat{N}\Delta H\hat{G}. \end{aligned} \tag{3.3.27}$$

Let us focus on the last term, which, once expanded, becomes

$$\begin{aligned} &\hat{G}\left[t_F\sum_x \psi_{x+1}^\dagger\psi_x + t_B\sum_{x\ \text{even}} b_{x+1}^{1\dagger}b_x^1 + t_B\sum_{x\ \text{odd}} b_{x+1}^{2\dagger}b_x^2 + \text{H.c.}\right] \\ &\times \hat{N}\frac{1}{U\sum_z (G_z)^2}\hat{N} \\ &\times \left[t_F\sum_y \psi_{y+1}^\dagger\psi_y + t_B\sum_{y\ \text{even}} b_{y+1}^{1\dagger}b_y^1 + t_B\sum_{y\ \text{odd}} b_{y+1}^{2\dagger}b_y^2 + \text{H.c.}\right]\hat{G}. \end{aligned} \tag{3.3.28}$$

Observe that we are allowed to consider the inverse of the operator in the above expression since in the non gauge invariant states subspace $U\sum_z (G_z)^2$ has no vanishing eigenvalues. To compute the matrix elements of (3.3.28) let us first consider the gauge invariant state $|\Phi_G\rangle = \hat{G}|\Phi\rangle$, and the following operator acting on it:

$$\left[t_F\sum_y \psi_{y+1}^\dagger\psi_y + t_B\sum_{y\ \text{even}} b_{y+1}^{1\dagger}b_y^1 + t_B\sum_{y\ \text{odd}} b_{y+1}^{2\dagger}b_y^2 + \text{H.c.}\right]|\Phi_G\rangle. \tag{3.3.29}$$

In this a superposition of states let us select for definiteness one of these vector, $b_{x+1}^{1\dagger}b_x^1|\Phi_G\rangle$, and study the quantity

$$U\sum_z (G_z)^2 b_{x+1}^{1\dagger}b_x^1|\Phi_G\rangle. \tag{3.3.30}$$

We have that

$$G_z|\Phi_G\rangle = \begin{cases} 0 & \text{if } z \neq x,\ z \neq x+1 \\ (-1)|\Phi_G\rangle & \text{if } z = x+1 \\ (+1)|\Phi_G\rangle & \text{if } z = x \end{cases}, \tag{3.3.31}$$

and therefore it follows

$$U\sum_z (G_z)^2 b^{1\dagger}_{x+1} b^1_x |\Phi_G\rangle = 2U|\Phi_G\rangle. \tag{3.3.32}$$

This result is independent from the choice of the point and the hopping term which acts on the state, and applies to each state projected on $\mathcal{H}_G$; by taking the inverse of the operator in (3.3.32) we can simplify the expression (3.3.28) by replacing the operator H^{-1}_{NN} with the factor $1/2U$:

$$\begin{aligned}
&\frac{1}{2U}\hat{G}\left[t_F\sum_x \psi^\dagger_{x+1}\psi_x + t_B\sum_{x\ \text{even}} b^{1\dagger}_{x+1}b^1_x + t_B\sum_{x\ \text{odd}} b^{2\dagger}_{x+1}b^2_x + \text{H.c.}\right]\\
&\quad\times\left[t_F\sum_y \psi^\dagger_{y+1}\psi_y + t_B\sum_{y\ \text{even}} b^{1\dagger}_{y+1}b^1_y + t_B\sum_{y\ \text{odd}} b^{2\dagger}_{y+1}b^2_y + \text{H.c.}\right]\hat{G}\\
&=\frac{1}{2U}\hat{G}\left[t_F\sum_x \psi^\dagger_{x+1}\psi_x + t_B\sum_{x\ \text{even}} U^1_{x,x+1} + t_B\sum_{x\ \text{odd}} U^2_{x,x+1} + \text{H.c.}\right]\\
&\quad\times\left[t_F\sum_y \psi^\dagger_{y+1}\psi_y + t_B\sum_{y\ \text{even}} U^1_{y,y+1} + t_B\sum_{y\ \text{odd}} U^2_{y,y+1} + \text{H.c.}\right]\hat{G}.
\end{aligned} \tag{3.3.33}$$

The final step is to select the non vanishing terms of the above expression: in the following we will expand the product term by term. Let us first consider

$$\begin{aligned}
&\frac{1}{2U}\hat{G}\left[t_F\sum_x \psi^\dagger_{x+1}\psi_x + \text{H.c.}\right]\left[t_F\sum_y \psi^\dagger_{y+1}\psi_y + \text{H.c.}\right]\hat{G}\\
&=\frac{t_F^2}{2U}\sum_x\sum_y\left[\psi^\dagger_{x+1}\psi_x\psi^\dagger_y\psi_{y+1}\delta_{x\,y} + \psi^\dagger_x\psi_{x+1}\psi^\dagger_{y+1}\psi_y\delta_{x\,y}\right]\\
&=\frac{t_F^2}{2U}\sum_x\left[\psi^\dagger_{x+1}\psi_x\psi^\dagger_x\psi_{x+1} + \psi^\dagger_x\psi_{x+1}\psi^\dagger_{x+1}\psi_x\right]\\
&=\frac{t_F^2}{2U}\sum_x\left[\psi^\dagger_{x+1}(1-\psi^\dagger_x\psi_x)\psi_{x+1} + \psi^\dagger_x\psi_{x+1}\psi^\dagger_{x+1}\psi_x\right]\\
&=\frac{t_F^2}{2U}\sum_x\left[\psi^\dagger_{x+1}\psi_{x+1} - \psi^\dagger_x\psi_x\psi^\dagger_{x+1}\psi_{x+1} - \psi^\dagger_x\psi_x\psi^\dagger_{x+1}\psi_{x+1} + \psi^\dagger_x\psi_x\right]\\
&=\frac{t_F^2}{2U}\sum_x\left[n^F_{x+1} - n^F_x n^F_{x+1} - n^F_x n^F_{x+1} + n^F_x\right]\\
&=\frac{t_F^2}{2U}\sum_x\left[(n^F_{x+1})^2 - 2n^F_x n^F_{x+1} + (n^F_x)^2\right]\\
&=\frac{t_F^2}{2U}\sum_x\left[n^F_{x+1} - n^F_x\right]^2.
\end{aligned} \tag{3.3.34}$$

A key observation to justify the first equality in the above result is that a given state $|\Phi\rangle$ is first projected on $\mathcal{H}_G$; then its gauge invariance is broken by the action of the terms in the first square brackets from the right. After that, terms in the second square brackets modify the state, and the final result is projected again on $\mathcal{H}_G$. It follows that the only terms which are not annihilated by $\hat{G}$, in all those arising from the product of the two brackets, are the gauge invariant ones.

We have already discussed the structure of a gauge invariant hopping operator: we deduced that it must contain four field operators acting on two sites. Also, on each site a creation and a destruction field operator must act, in order to preserve condition (3.2.10). This reasoning applies to all the other terms of the product (3.3.33), which are computed in the following. The next product we consider is

$$\begin{aligned}
&\frac{1}{2U}\hat{G}\left[t_F\sum_x \psi^\dagger_{x+1}\psi_x + \text{H.c.}\right]\left[t_B\sum_{y\ \text{even}} b^{1\dagger}_{y+1}b^1_y + \text{H.c.}\right]\hat{G}\\
&= \frac{t_F t_B}{2U}\sum_x\sum_{y\ \text{even}}\left[\psi^\dagger_{x+1}\psi_x b^{1\dagger}_y b^1_{y+1}\delta_{x\,y} + \psi^\dagger_x\psi_{x+1}b^{1\dagger}_{y+1}b^1_y\delta_{x\,y}\right]\\
&= \frac{t_F t_B}{2U}\sum_{x\ \text{even}}\left[\psi^\dagger_{x+1}\psi_x b^{1\dagger}_x b^1_{x+1} + \psi^\dagger_x\psi_{x+1}b^{1\dagger}_{x+1}b^1_x\right]\\
&= \frac{t_F t_B}{2U}\sum_{x\ \text{even}}\left[\psi^\dagger_{x+1}U^{1\dagger}_{x,x+1}\psi_x + \psi^\dagger_x U^1_{x,x+1}\psi_{x+1}\right],
\end{aligned} \tag{3.3.35}$$

and similarly

$$\begin{aligned}
&\frac{1}{2U}\hat{G}\left[t_F\sum_x \psi^\dagger_{x+1}\psi_x + \text{H.c.}\right]\left[t_B\sum_{y\ \text{odd}} b^{2\dagger}_{y+1}b^2_y + \text{H.c.}\right]\hat{G}\\
&= \frac{t_F t_B}{2U}\sum_{x\ \text{odd}}\left[\psi^\dagger_{x+1}U^{2\dagger}_{x,x+1}\psi_x + \psi^\dagger_x U^2_{x,x+1}\psi_{x+1}\right].
\end{aligned} \tag{3.3.36}$$

By summing the results of (3.3.36) and (3.3.35), and observing that identical results follow from the product

$$\frac{1}{2U}\hat{G}\left[t_B\sum_{x\ \text{even}} b^{1\dagger}_{x+1}b^1_x + t_B\sum_{x\ \text{odd}} b^{2\dagger}_{x+1}b^2_x + \text{H.c.}\right]\left[t_F\sum_y \psi^\dagger_{y+1}\psi_y + \text{H.c.}\right]\hat{G}, \tag{3.3.37}$$

we obtain the term

$$\frac{t_F t_B}{U}\sum_x\left[\psi^\dagger_{x+1}U^\dagger_{x,x+1}\psi_x + \psi^\dagger_x U_{x,x+1}\psi_{x+1}\right], \tag{3.3.38}$$

in which we have omitted for compactness the indices σ on $U_{x,x+1}$ relative to the site parity. Now we consider the products between the comparators. Let us

first observe that

$$\begin{aligned}
&\frac{1}{2U}\hat{G}\left[t_B\sum_{x\ \text{odd}}U^2_{x,x+1}+\text{H.c.}\right]\left[t_B\sum_{y\ \text{even}}U^1_{y,y+1}+\text{H.c.}\right]\hat{G}\\
=&\frac{1}{2U}\hat{G}\left[t_B\sum_{x\ \text{even}}U^1_{x,x+1}+\text{H.c.}\right]\left[t_B\sum_{y\ \text{odd}}U^2_{y,y+1}+\text{H.c.}\right]\hat{G}=0,
\end{aligned}\tag{3.3.39}$$

since we are multiplying comparators acting on different links and the terms we obtain from the product are not gauge invariant.

The last products to be computed are those products between the comparators on the same link: for $\sigma = 1$ we have

$$\begin{aligned}
&\frac{1}{2U}\hat{G}\left[t_B\sum_{x\ \text{even}}U^1_{x,x+1}+\text{H.c.}\right]\left[t_B\sum_{y\ \text{even}}U^1_{y,y+1}+\text{H.c.}\right]\hat{G}\\
=&\frac{t_B^2}{2U}\sum_{x\ \text{even}}\left[U^1_{x,x+1}U^{1\dagger}_{x,x+1}+U^{1\dagger}_{x,x+1}U^1_{x,x+1}\right]\\
=&\frac{t_B^2}{2U}\sum_{x\ \text{even}}\left[b^{1\dagger}_{x+1}b^1_x b^{1\dagger}_x b^1_{x+1}+b^{1\dagger}_x b^1_{x+1}b^{1\dagger}_{x+1}b^1_x\right]\\
=&\frac{t_B^2}{2U}\sum_{x\ \text{even}}\left[b^{1\dagger}_{x+1}b^1_{x+1}(1+b^{1\dagger}_x b^1_x)+b^{1\dagger}_x b^1_x(1+b^{1\dagger}_{x+1}b^1_{x+1})\right]\\
=&\frac{t_B^2}{2U}\sum_{x\ \text{even}}\left[2b^{1\dagger}_{x+1}b^1_{x+1}b^{1\dagger}_x b^1_x+2S\right]=\frac{t_B^2}{U}\sum_{x\ \text{even}}\left[b^{1\dagger}_{x+1}b^1_{x+1}b^{1\dagger}_x b^1_x+S\right],
\end{aligned}\tag{3.3.40}$$

in which we have used the commutators (3.2.1) and the condition that $n^1_x + n^1_{x+1} = 2S$. Similarly, for $\sigma = 2$ it follows that

$$\begin{aligned}
&\frac{1}{2U}\hat{G}\left[t_B\sum_{x\ \text{odd}}U^2_{x,x+1}+\text{H.c.}\right]\left[t_B\sum_{y\ \text{odd}}U^2_{y,y+1}+\text{H.c.}\right]\hat{G}\\
=&\frac{t_B^2}{U}\sum_{x\ \text{odd}}\left[b^{2\dagger}_{x+1}b^2_{x+1}b^{2\dagger}_x b^2_x+S\right],
\end{aligned}\tag{3.3.41}$$

and the results (3.3.40) and (3.3.41) can be summed omitting the constant terms, yielding the following

$$\frac{t_B^2}{U}\left[\sum_{x\ \text{even}}\left[b^{1\dagger}_{x+1}b^1_{x+1}b^{1\dagger}_x b^1_x\right]+\sum_{x\ \text{odd}}\left[b^{2\dagger}_{x+1}b^2_{x+1}b^{2\dagger}_x b^2_x\right]\right].\tag{3.3.42}$$

Now let us extract a step from calculation (3.3.11), since it will lead us to a

helpful result. Let us observe that

$$\begin{aligned}
&\sum_x (E_{x,x+1})^2 \overset{(3.3.11)}{=} \\
&= \sum_{x \text{ even}} \frac{1}{4}[(n^1_{x,x+1}) - 2n^1_{x+1}n^1_x + (n^1_x)^2] + \sum_{x \text{ odd}} \frac{1}{4}[(n^2_{x,x+1})^2 - 2n^2_{x+1}n^2_x + (n^2_x)^2] \\
&= \sum_{x \text{ even}} \frac{1}{4}[-4n^1_{x+1}n^1_x + 4S^2] + \sum_{x \text{ odd}} \frac{1}{4}[-4n^2_{x+1}n^2_x + 4S^2] \\
&= \sum_{x \text{ even}} [-n^1_{x+1}n^1_x + 4S^2] + \sum_{x \text{ odd}} [-n^2_{x+1}n^2_x + 4S^2], \qquad (3.3.43)
\end{aligned}$$

since

$$-2n^\sigma_{x+1}n^\sigma_x = -(n^\sigma_{x+1})^2 - (n^\sigma_x)^2 + 4S^2,$$

therefore up to constant terms we can write

$$\frac{t_B^2}{U}\sum_x (E_{x,x+1})^2 = -\frac{t_B^2}{U}\sum_{x \text{ even}} [n^1_{x+1}n^1_x] - \frac{t_B^2}{U}\sum_{x \text{ odd}} [n^2_{x+1}n^2_x]. \qquad (3.3.44)$$

We now can put together all terms and write [2]

$$\begin{aligned}
H_{eff} =& \frac{g^2}{2}\sum_x (E_{x,x+1})^2 - \frac{t_F t_B}{U}\sum_x \left[\psi^\dagger_{x+1}U^\dagger_{x,x+1}\psi_x + \text{H.c.}\right] \\
&- \frac{t_B^2}{U}\left[\sum_{x \text{ even}} \left[b^{1\dagger}_{x+1}b^1_{x+1}b^{1\dagger}_x b^1_x\right] + \sum_{x \text{ odd}} \left[b^{2\dagger}_{x+1}b^2_{x+1}b^{2\dagger}_x b^2_x\right]\right] \\
&+ m\sum_x (-1)^x n^F_x - \frac{t_F^2}{2U}\sum_x \left[n^F_{x+1} - n^F_x\right]^2 \\
=& -\frac{t_F t_B}{U}\sum_x \left[\psi^\dagger_{x+1}U^\dagger_{x,x+1}\psi_x + \text{H.c.}\right] + \left(\frac{g^2}{2} + \frac{t_B^2}{U}\right)\sum_x (E_{x,x+1})^2 \\
&+ m\sum_x (-1)^x n^F_x - \frac{t_F^2}{2U}\sum_x \left[n^F_{x+1} - n^F_x\right]^2. \qquad (3.3.45)
\end{aligned}$$

Therefore we obtain the effective Hamiltonian (3.3.27) if we recall the coupling constants

$$\frac{(g')^2}{2} = \left(\frac{g^2}{2} + \frac{t_B^2}{U}\right), \qquad t = \frac{t_F t_B}{U}.$$

Note finally that we have obtained another gauge invariant term containing the fermion number operators, but we can limit its influence with a suitable choice of the parameters t_B, t_F, U related to the physical system.

Numerical simulations have been performed with a four-site lattice and $S = 1$, taking $U \sim 10t_F$, and $U \ll t_B$. The probability for a gauge invariant state to evolve out of $\mathcal{H}_G$ was $< 10\%$, in a time interval $\tau \sim 5000/t$; a similar simulation with $U \sim 20t_F$ yielded a gauge invariance-breaking probability $< 2\%$ [2].

Chapter 4

The Weyl group and its application to define a unitary comparator

In this chapter we define a unitary comparator for our model. In the first section we introduce the continuous Weyl group and the Schrödinger representation, discussing some properties of the group. In the second section the discrete Weyl group is constructed and some analogies and differences with the continuous group are analyzed. In the last section the unitary comparator is defined, on the basis of concepts and tools introduced in the preceding sections.

4.1 A remark on the Quantum Link Model and an outlook

In Chapter 3 we have used the Quantum Link model to pass from a continuous electric field to a finite, discrete one, defined in (3.1.5), and in this way from an infinite dimensional lattice link Hilbert space to a finite one. On this finite space we have defined in (3.1.6) a comparator satisfying the same algebra commutator (3.1.8) with the electric field as in the continuous field case: due to this relation generators (3.1.10) commute with the Hamiltonian (3.1.9), and the implementation of a gauge theory has been possible. A problem arises anyway, since the comparator $U_{QLM\,x,x+1}$ is not unitary. Also, it could be shown that it is impossible to define, on a finite Hilbert space, a unitary operator $\hat{A}$ and a Hermitian operator $\hat{B}$ satisfying the commutation rule

$$[\hat{B}, \hat{A}] = \hat{A}. \tag{4.1.1}$$

By considering the continuous electric field and the relative comparator, the above commutator descends, like in (2.2.8), from the algebra commutation rule (2.2.7) between the electric field and the vector potential. In this chapter we will show that in case of discrete fields this implication is not true; moreover, we will

find that there exist no group generators algebra. We will introduce therefore the formalism of the group commutators, in the continuous field case and then in the discrete one. Our goal is to define alternative quantities to implement a gauge theory on a lattice with discrete links Hilbert spaces, without renouncing to the definition of a unitary comparator.

4.2 The Weyl group

4.2.1 The commutator and the abstract group

In this section we will introduce the continuous Weyl group: we will first construct the group commutator by studying the representation of the group on the Hilbert space of a quantum particle states, then will define the abstract group. In the second subsection a particular representation, the Schrödinger's one, will be discussed.

Let us start form the description of a quantum mechanical system consisting of a particle moving on a line; its degrees of freedom are the particle position q and momentum p, and $\mathcal{H}$ the system Hilbert space. We start from Heisenberg's algebra, to define Weyl's operators. Heisenberg associates two Hermitian operators to the particle's position and momentum, $\hat{p}$ and $\hat{q}$, obeying the commutation rules [20]

$$[\hat{q}, \hat{p}] = i, \qquad [\hat{q}, \hat{q}] = 0, \qquad [\hat{p}, \hat{p}] = 0, \tag{4.2.1}$$

in which $\hbar = 1$. Operators $\hat{p}$ and $\hat{q}$ and their polynomial functions form a representation of Heisenberg's algebra, whose composition law is defined by (4.2.1).

We interpret operators $\hat{p}$ and $\hat{q}$ as generators of infinitesimal unitary transformations of vectors in $\mathcal{H}$: these transformations are

$$\delta U = \mathbb{1} + i\frac{\sigma_q}{N}\hat{p}, \qquad\qquad \delta V = \mathbb{1} + i\frac{\sigma_p}{N}\hat{q}, \tag{4.2.2}$$

in which σ_q and σ_p are real and finite parameters, and N a large positive integer. We obtain finite transformations by taking the limit

$$\lim_{N\to\infty}\left(1 + i\frac{\sigma_q\hat{p} + \sigma_p\hat{q}}{N}\right)^N. \tag{4.2.3}$$

In conclusion we define Weyl's operators:

$$W(\sigma_q, \sigma_p) = \mathrm{e}^{i(\sigma_q\hat{p} + \sigma_p\hat{q})}, \tag{4.2.4}$$

$$U(\sigma_q) = \mathrm{e}^{i\sigma_q\hat{p}}, \tag{4.2.5}$$

$$V(\sigma_p) = \mathrm{e}^{i\sigma_p\hat{q}}. \tag{4.2.6}$$

The choice of the label q in the parameter σ_q multiplying $\hat{p}$ and viceversa will be clear later. Since $\hat{p}$ and $\hat{q}$ do not commute $W(\sigma_q, \sigma_p) \neq U(\sigma_q)V(\sigma_p)$, and the

following relation holds:

$$W(\sigma_q, \sigma_p) = U(\sigma_q)V(\sigma_p)\mathrm{e}^{\frac{1}{2}\sigma_q\sigma_p[\hat{p},\hat{q}]}. \tag{4.2.7}$$

This result is an immediate consequence of the Baker-Hausdorff formula, which states that

$$\mathrm{e}^{A+B} = \mathrm{e}^{A}\mathrm{e}^{B}\mathrm{e}^{-\frac{1}{2}[A,B]}, \tag{4.2.8}$$

if $C = [A, B]$ commutes with A and B. It follows that

$$\begin{aligned} \mathrm{e}^{B}\mathrm{e}^{A}\mathrm{e}^{-\frac{1}{2}[B,A]} &= \mathrm{e}^{A}\mathrm{e}^{B}\mathrm{e}^{-\frac{1}{2}[A,B]} \quad \Rightarrow \\ \mathrm{e}^{B}\mathrm{e}^{A}\mathrm{e}^{\frac{1}{2}[A,B]} &= \mathrm{e}^{A}\mathrm{e}^{B}\mathrm{e}^{-\frac{1}{2}[A,B]} \quad \Rightarrow \\ \mathrm{e}^{[A,B]} &= \mathrm{e}^{-A}\mathrm{e}^{-B}\mathrm{e}^{A}\mathrm{e}^{B}, \end{aligned} \tag{4.2.9}$$

therefore the group commutator between $U(\sigma_q)$ and $V(\sigma_p)$ is

$$V^{\dagger}(\sigma_p)U^{\dagger}(\sigma_q)V(\sigma_p)U(\sigma_q) = \mathrm{e}^{-i\sigma_p\sigma_q}. \tag{4.2.10}$$

Now that we have obtained the commutator between the Weyl operators in a given representation, we can define the abstract Weyl group. The abstract Weyl group is a two-real and continuous parameters group whose generators obey the Heisenberg's algebra relation. Let us define the group elements $\tilde{W}(\sigma_q, \sigma_p)$, $\tilde{U}(\sigma_q)$ and $\tilde{V}(\sigma_p)$ whose commutators are fixed by (4.2.7) and (4.2.10):

$$\tilde{V}^{\dagger}(\sigma_p)\tilde{U}^{\dagger}(\sigma_q)\tilde{V}(\sigma_p)\tilde{U}(\sigma_q) = \mathrm{e}^{-i\sigma_p\sigma_q}, \tag{4.2.11}$$

$$\tilde{W}(\sigma_q, \sigma_p) = \tilde{U}(\sigma_q)\tilde{V}(\sigma_p)\mathrm{e}^{\frac{1}{2}\sigma_q\sigma_p[\hat{p},\hat{q}]}. \tag{4.2.12}$$

These elements and all their compositions form the abstract Weyl group.

4.2.2 A few comments on the Schrödinger representation of the Weyl group

From the Von Neumann theorem we know that *all irreducible representations of the Weyl group, such that unitary operators representing $U(\sigma_q)$ and $V(\sigma_p)$ are strongly continuous in σ_p and σ_q, respectively, are unitarily equivalent* [21]: that is, given two irreducible representations satisfying the continuity condition, there exists a unitary transformation to pass from one to the other. Therefore it is sufficient to know one representation of the Weyl group: the representation we will examine is the Schrödinger one. The Hilbert space is

$$\mathcal{H} = \left\{ \psi \in L^2(\mathbb{R}, dx) \;\middle|\; \|\psi\|^2 = \int_{-\infty}^{+\infty} |\psi(x)|^2 \,\mathrm{d}x = 1 \right\}, \tag{4.2.13}$$

and its elements are called wave functions. For simplicity we will indicate the Weyl operators and the Hilbert space of this representation with the same notation we used in the previous subsection. Given a wave function $\psi(x) \in \mathcal{H}$ the

group is represented as follows:

$$U(\sigma_q)\psi(x) = \mathrm{e}^{i\sigma_q\hat{p}}\psi(x) = \psi(x+\sigma_q) \tag{4.2.14}$$

$$V(\sigma_p)\psi(x) = \mathrm{e}^{i\sigma_p\hat{q}}\psi(x) = \mathrm{e}^{i\sigma_p x}\psi(x). \tag{4.2.15}$$

The representation can be shown to be unitary, strongly continuous and irreducible, so Von Neumann's theorem hypothesis are satisfied.
Some comments are needed. First, given the infinitesimal parameters ϵ_q and ϵ_p, and taken ψ in the test function space $\in \mathcal{S}(\mathbb{R})$, by applying (4.2.14) one has

$$\begin{aligned} U(\epsilon_q)\psi(x) &= \mathrm{e}^{i\epsilon_q\hat{p}}\psi(x) \\ &= (\mathbb{1} + i\epsilon_q\hat{p} + O(\epsilon_q^2))\psi(x) \\ &= \psi(x+\epsilon_q) \\ &= \psi(x) + \epsilon_q\frac{d}{dx}\psi(x) + O(\epsilon_q^2), \end{aligned} \tag{4.2.16}$$

$$\begin{aligned} V(\epsilon_p)\psi(x) &= \mathrm{e}^{i\sigma_p\hat{q}}\psi(x) \\ &= (\mathbb{1} + i\epsilon_p\hat{q} + O(\epsilon_p^2))\psi(x) \\ &= \mathrm{e}^{i\epsilon_p x}\psi(x) \\ &= \psi(x) + i\epsilon_p x\psi(x) + O(\epsilon_p^2). \end{aligned} \tag{4.2.17}$$

It follows that

$$\hat{q}\psi(x) = x\psi(x) \tag{4.2.18}$$

$$\hat{p}\psi(x) = -i\frac{d}{dx}\psi(x). \tag{4.2.19}$$

Let us remark that in relations (4.2.16)...(4.2.19) we have chosen $\psi(x)$ in the test functions set $\mathcal{S}(\mathbb{R})$: on this domain we can appropriately define the Heisenberg's algebra generated from $\hat{p}$ and $\hat{q}$. The existence of generators is guaranteed by Stone's theorem, since we are dealing with a strongly continuous representation, defined on a dense set of the Hilbert space [22].

Finally, note that as long as the wave functions are represented in the position basis, $\hat{q}$ is a multiplicative operator and the operator $\hat{p}$ returns the derivative of the wave function. Now we will examine the action of these operators if the wave functions are expressed in the momentum basis, where the state vector is represented by the Fourier transform of $\psi(x) \in \mathcal{H}$. In this basis the expectation values of position and momentum can be evaluated as follows on a given state $\psi \in \mathcal{S}(\mathbb{R})$:

$$\begin{aligned} \langle\hat{p}\rangle_\psi &= -i\int_{-\infty}^{+\infty}\psi^*(x)\frac{\mathrm{d}}{\mathrm{d}x}\psi(x)\,\mathrm{d}x \\ &= -i\frac{1}{2\pi}\iiint_{-\infty}^{+\infty}\tilde{\psi}^*(k')\mathrm{e}^{-ik'x}\tilde{\psi}(k)\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{ikx}\,\mathrm{d}x\,\mathrm{d}k\,\mathrm{d}k' = \end{aligned}$$

$$= \int_{-\infty}^{+\infty} \tilde{\psi}^*(k')\tilde{\psi}(k)k\delta(k-k')\,\mathrm{d}k\,\mathrm{d}k' \tag{4.2.20}$$

$$= \int_{-\infty}^{+\infty} k|\tilde{\psi}(k)|^2\,\mathrm{d}k, \tag{4.2.21}$$

while for position we have

$$\begin{aligned}
\langle\hat{q}\rangle_\psi =& \int_{-\infty}^{+\infty} \psi^*(x)x\psi(x)\,\mathrm{d}x \\
=& \frac{1}{2\pi}\iiint \tilde{\psi}^*(k')\mathrm{e}^{-ik'x}x\mathrm{e}^{+ikx}\tilde{\psi}(k)\,\mathrm{d}x\,\mathrm{d}k\,\mathrm{d}k' \\
=& -i\frac{1}{2\pi}\iiint \tilde{\psi}^*(k')\mathrm{e}^{-ik'x}\tilde{\psi}(k)\frac{\mathrm{d}}{\mathrm{d}k}\mathrm{e}^{+ikx}\,\mathrm{d}x\,\mathrm{d}k\,\mathrm{d}k' \\
=& -i\frac{1}{2\pi}\underbrace{\iint \tilde{\psi}^*(k')\mathrm{e}^{-ik'x}\mathrm{e}^{+ikx}\tilde{\psi}(k)\,\mathrm{d}x\,\mathrm{d}k'\Big|_{k=-\infty}^{+\infty}}_{0} \\
&+ i\frac{1}{2\pi}\iiint \tilde{\psi}^*(k')\mathrm{e}^{-ik'x}\mathrm{e}^{+ikx}\frac{\mathrm{d}}{\mathrm{d}k}\tilde{\psi}(k)\,\mathrm{d}x\,\mathrm{d}k'\,\mathrm{d}k \\
=& i\int_{-\infty}^{+\infty} \tilde{\psi}^*(k)\frac{\mathrm{d}}{\mathrm{d}k}\tilde{\psi}(k)\,\mathrm{d}k.
\end{aligned} \tag{4.2.22}$$

We deduce that in the momentum basis

$$\hat{q}\tilde{\psi}(k) = i\frac{\mathrm{d}}{\mathrm{d}k}\tilde{\psi}(k), \tag{4.2.23}$$

$$\hat{p}\tilde{\psi}(k) = k\tilde{\psi}(k), \tag{4.2.24}$$

so the action of the operators on wave functions has been inverted with respect to the definition given in (4.2.19). This symmetry emerges also in the representation of $V(\sigma_p)$ and $U(\sigma_q)$: from (4.2.23) it follows that

$$\begin{aligned}
U(\sigma_q)\tilde{\psi}(k) &= \mathrm{e}^{i\sigma_q\hat{p}}\tilde{\psi}(k) = \mathrm{e}^{i\sigma_q k}\tilde{\psi}(k), \\
V(\sigma_p)\tilde{\psi}(k) &= \mathrm{e}^{i\sigma_p\hat{q}}\tilde{\psi}(k) = \tilde{\psi}(k-\sigma_p).
\end{aligned} \tag{4.2.25}$$

In conclusion we have defined the Weyl group starting from the Heisenberg operators $\hat{p}$ and $\hat{q}$. We have deduced the commutators on Weyl's group from Heisenberg's algebra relations. Finally we have considered the Schrödinger representation of the Weyl group. Through this, we have shown the action of operators $\hat{p}$ and $\hat{q}$ on the wave functions expressed in position and momentum basis, and we have stressed the specular results (4.2.14) and (4.2.25). Analogue relations will be found in the next section, in which the discrete Weyl group will be constructed.

4.3 The discrete Schwinger-Weyl group

The construction of the discrete Weyl group is different from that of the continuous one, but, *mutatis mutandis*, similar results obtained in the previous section will be deduced in this case as well [23].
Let us consider an n-dimensional Hilbert space $\mathcal{H}$ and choose an orthonormal basis of vectors

$$\{|a_l\rangle\}_{1\leq l\leq n}, \quad \text{with} \quad \langle a_l|a_{l'}\rangle = \delta_{ll'}. \tag{4.3.1}$$

Let us define a unitary operator U with these features:

$$U|a_l\rangle = |a_{l+1}\rangle, \tag{4.3.2}$$
$$U|a_n\rangle = |a_1\rangle, \tag{4.3.3}$$
$$U^n = \mathbb{1}. \tag{4.3.4}$$

Condition (4.3.3) in particular makes it unitary and distinguishes U from a ladder operator U', for which one would have

$$U'|a_l\rangle = c_l|a_{l+1}\rangle, \tag{4.3.5}$$
$$U'|a_n\rangle = 0, \tag{4.3.6}$$
$$U'^{\dagger}|a_1\rangle = 0, \tag{4.3.7}$$

where c_l is a normalization factor. In the basis we have chosen the form of U is:

$$U = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.3.8}$$

If we call $\{|u_k\rangle\}_{1\leq k\leq n}$ the eigenvectors of U, and $\{u_k\}_{1\leq k\leq n}$ their eigenvalues, they satisfy the equation

$$U|u_k\rangle = u_k|u_k\rangle,$$

from which it follows that

$$\begin{aligned} U^n|u_k\rangle &= u_k^n|u_k\rangle \\ &= |u_k\rangle. \end{aligned} \tag{4.3.9}$$

We want to express the eigenvectors $\{|u_k\rangle\}_{1\leq k\leq n}$ as linear combinations of the basis $\{|a_l\rangle\}_{1\leq l\leq n}$, and to do this we need a number of intermediate steps. From (4.3.9) it follows that eigenvalues u_k of U are the n solutions of the equation $x^n = 1$, namely

$$u_k = \mathrm{e}^{\frac{2\pi i}{n}k}, \qquad 1 \leq k \leq n. \tag{4.3.10}$$

By writing U in the form

$$U = \sum_{k=1}^{n} u_k |u_k\rangle\langle u_k|, \tag{4.3.11}$$

we can obtain an expression for the projector $|u_k\rangle\langle u_k|$, with k arbitrarily taken in $\{1, \ldots, n\}$. Let us first observe that

$$\begin{aligned} U^n - \mathbb{1} &= \left(\frac{U}{u_k}\right)^n - \mathbb{1} = \left(\frac{U}{u_k} - \mathbb{1}\right) \sum_{l=0}^{n-1} \left(\frac{U}{u_k}\right)^l = \left(\frac{U}{u_k} - \mathbb{1}\right) \sum_{l=1}^{n} \left(\frac{U}{u_k}\right)^l \\ &= \sum_{j=1}^{n} \left(\frac{u_j}{u_k} |u_j\rangle\langle u_j| - |u_j\rangle\langle u_j|\right) \sum_{l=1}^{n} \sum_{i=1}^{n} \left(\frac{u_i}{u_k}\right)^l |u_i\rangle\langle u_i| \\ &= \sum_{j=1}^{n} \left(\frac{u_j}{u_k} - 1\right) \sum_{l=1}^{n} \left(\frac{u_j}{u_k}\right)^l |u_j\rangle\langle u_j| = 0 \qquad \forall k. \end{aligned} \tag{4.3.12}$$

where we have used (4.3.11) to replace U. We have obtained the sum of n independent projectors, therefore each coefficient labelled with j must vanish. For $j = k$ the sum over l is n and the factor $\left(\frac{u_j}{u_k} - 1\right)$ vanishes; for $j \neq k$ it results $\left(\frac{u_j}{u_k} - 1\right) \neq 0$ and the sum over l must be zero. It follows that

$$\sum_{l=1}^{n} \left(\frac{u_j}{u_k}\right)^l = n\delta_{jk}, \qquad \forall j, k \in \{1, \ldots, n\} \tag{4.3.13}$$

and then the final result is

$$\begin{aligned} \frac{1}{n} \sum_{l=1}^{n} \left(\frac{U}{u_k}\right)^l &= \frac{1}{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \left(\frac{u_j}{u_k}\right)^l |u_j\rangle\langle u_j| \\ &= |u_k\rangle\langle u_k|, \qquad\qquad 1 \leq k \leq n \end{aligned} \tag{4.3.14}$$

which is the relation we were looking for.

The next step is to consider the action of projector $|u_k\rangle\langle u_k|$ on the basis vector $|a_n\rangle$: we have

$$\begin{aligned} |u_k\rangle\langle u_k|a_n\rangle &= \frac{1}{n} \sum_{l=1}^{n} \left(\frac{U}{u_k}\right)^l |a_n\rangle = \frac{1}{n} \sum_{l=1}^{n} \frac{U^{l+n-1}}{(u_k)^l} |a_1\rangle = \frac{1}{n} \sum_{l=1}^{n} \frac{U^{l-1}}{(u_k)^l} |a_1\rangle \\ &= \frac{1}{n} \sum_{l=1}^{n} \frac{|a_l\rangle}{(u_k)^l}, \end{aligned} \tag{4.3.15}$$

and multiplying from left by the bra $\langle a_n|$ it results

$$|\langle a_n|u_k\rangle|^2 = \frac{1}{n}. \tag{4.3.16}$$

Without loss of generality let us fix to one the arbitrary phase and assume that

$$\langle u_k|a_n\rangle = \frac{1}{\sqrt{n}}. \tag{4.3.17}$$

By multiplying the quantity in (4.3.15) from the left by the generic bra $\langle a_l|$, one gets

$$\begin{aligned}\langle a_l|u_k\rangle\langle u_k|a_n\rangle &= \frac{1}{n}\langle a_l|\sum_{l'=1}^{n}\frac{|a_{l'}\rangle}{u_k^{l'}} \\ &= \frac{1}{n}\mathrm{e}^{-\frac{2\pi ik}{n}l},\end{aligned} \tag{4.3.18}$$

which implies

$$\langle a_l|u_k\rangle = \frac{1}{\sqrt{n}}\mathrm{e}^{-\frac{2\pi ik}{n}l}. \tag{4.3.19}$$

So finally we can write

$$\begin{aligned}|u_k\rangle &= \sum_{l=1}^{n}\langle a_l|u_k\rangle|a_l\rangle \\ &= \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\mathrm{e}^{-\frac{2\pi ik}{n}l}|a_l\rangle.\end{aligned} \tag{4.3.20}$$

The eigenvectors of U are related to the basis vectors by the discrete Fourier transform.

So far we have defined the operator U, shown its form in a particular basis and written its eigenvalues and eigenvectors. We have found out that the eigenvectors of U are given by the discrete Fourier transform of the basis vectors: we have used the fact that U cyclically permutes vectors $\{|a_l\rangle\}_{1\leq l\leq n}$, according to the definition (4.3.2).
Now we will study a new operator, V, which permutes U's eigenvectors, defined as follows:

$$\langle u_k|V = \langle u_{k+1}| \tag{4.3.21}$$

$$\langle u_n|V = \langle u_1| \tag{4.3.22}$$

$$V^n = \mathbb{1}. \tag{4.3.23}$$

Eigenvalues of V coincide with those of U, namely

$$v_l = \mathrm{e}^{\frac{2\pi i}{n}l}, \qquad 1 \leq l \leq n, \tag{4.3.24}$$

and, with the same procedure used before, we can write the projection relative to the generic eigenvector $|v_l\rangle$ in the form

$$|v_l\rangle\langle v_l| = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{V}{v_l}\right)^k. \tag{4.3.25}$$

Since V has been defined on the dual space of $\mathcal{H}$, it is more convenient to study $\langle v_l|u_k\rangle$ by multiplying $|v_l\rangle\langle v_l|$ by $\langle u_n|$ from the left:

$$\begin{aligned}\langle u_n|v_l\rangle\langle v_l| &= \langle u_n|\frac{1}{n}\sum_{k=1}^{n}\left(\frac{V}{v_l}\right)^k \\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{\langle u_k|}{v_l^k},\end{aligned} \tag{4.3.26}$$

from which, fixing an arbitrary phase, we obtain

$$\langle u_n|v_l\rangle = \frac{1}{\sqrt{n}}. \tag{4.3.27}$$

By observing that

$$\langle u_n|v_l\rangle\langle v_l|u_k\rangle = \frac{1}{n}v_l^{-k} \tag{4.3.28}$$

it follows that

$$\langle v_l|u_k\rangle = \frac{1}{\sqrt{n}}v_l^{-k} = \frac{1}{\sqrt{n}}\mathrm{e}^{-\frac{2\pi i l}{n}k}. \tag{4.3.29}$$

All the results just obtained are valid for $1 \le l \le n$ and $1 \le k \le n$; by comparing (4.3.29) with (4.3.19) we conclude that

$$\begin{aligned}|v_l\rangle &= \sum_{k=1}^{n}\langle u_k|v_l\rangle|u_k\rangle = \sum_{k=1}^{n}\langle u_k|a_l\rangle|u_k\rangle \\ &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\mathrm{e}^{\frac{2\pi i l}{n}k}|u_k\rangle = |a_l\rangle, \qquad 1 \le l \le n,\end{aligned} \tag{4.3.30}$$

that is, the basis we have initially chosen coincides with V's eigenvectors. Note that these vectors are related to $\{|u_k\rangle\}$ by mean of the inverse Fourier transform.

Now we can determine element by element the matrix form of V in the initial basis (4.3.1):

$$V_{lm} = \langle a_l|V|a_m\rangle = \langle v_l|V|v_m\rangle = \delta_{lm}\mathrm{e}^{\frac{2\pi i l}{n}}. \tag{4.3.31}$$

E.g., for an arbitrarily fixed n,

$$V = \begin{pmatrix} \mathrm{e}^{\frac{2\pi i}{n}} & 0 & \cdots & 0 & 0 \\ 0 & \mathrm{e}^{\frac{2\pi i}{n}2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{\frac{2\pi i}{n}n-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.3.32}$$

By using the property $v_k = v_{k+n}$ we can rename the eigenvalues with an equivalent choice, which will result more convenient later:

$$\begin{aligned} v_{-n/2} = \mathrm{e}^{-\frac{2\pi i}{n}\frac{n}{2}}, \ldots, v_{n/2-1} = \mathrm{e}^{\frac{2\pi i}{n}\left(\frac{n}{2}-1\right)}, \qquad & n \quad \text{even}, \\ v_{-(n-1)/2} = \mathrm{e}^{-\frac{2\pi i}{n}\frac{n-1}{2}}, \ldots, v_{(n-1)/2} = \mathrm{e}^{\frac{2\pi i}{n}\frac{n-1}{2}}, \qquad & n \quad \text{odd}, \end{aligned} \tag{4.3.33}$$

and the same can be done for $\{u_k\}_{1\leq k\leq n}$.

Observe that we have constructed two operators, V and U, which have the same spectrum, since they satisfy property (4.3.4) and (4.3.23); also, by construction, each of them cyclically permutes the eigenvectors relative to the other operator, namely

$$\begin{aligned} U|u_k\rangle &= \mathrm{e}^{\frac{2\pi ik}{n}}|u_k\rangle & & \\ U|v_l\rangle &= |v_{l+1}\rangle, & |v_{l+n}\rangle &\equiv |v_l\rangle \\ V|v_l\rangle &= \mathrm{e}^{\frac{2\pi il}{n}}|v_l\rangle & & \\ \langle u_k|V &= \langle u_{k+1}|, & \langle u_{k+n}| &\equiv \langle u_k|. \end{aligned} \tag{4.3.34}$$

These relations are analogous to those obtained in the Schrödinger representation of the Weyl group, as one can verify by comparing them with (4.2.14) and (4.2.23). Other similarities between the continuous and discrete case can be found, but also some differences will be put on evidence.

Now we want to derive the group commutator. Operators U and V transform $|u_k\rangle$ as it follows,

$$\begin{aligned} \langle u_k|VU &= \langle u_{k+1}|U = \mathrm{e}^{\frac{2\pi i(k+1)}{n}}\langle u_{k+1}|, \\ \langle u_k|UV &= \mathrm{e}^{\frac{2\pi ik}{n}}\langle u_k|V = \mathrm{e}^{\frac{2\pi ik}{n}}\langle u_{k+1}|. \end{aligned} \tag{4.3.35}$$

Since $\{|u_k\rangle\}$ form a basis, relations (4.3.35) hold for all vectors in $\mathcal{H}$, therefore one obtains

$$VU = \mathrm{e}^{\frac{2\pi i}{n}}UV. \tag{4.3.36}$$

This relation can be generalized, in fact if we can consider

$$\begin{aligned} \langle u_j|V^kU^l &= \langle u_{j+k}|U^l = \mathrm{e}^{\frac{2\pi i(j+k)}{n}l}\langle u_{j+k}|, \\ \langle u_k|U^lV^k &= \mathrm{e}^{\frac{2\pi ik}{n}l}\langle u_k|V = \mathrm{e}^{\frac{2\pi ik}{n}l}\langle u_{j+k}|, \end{aligned} \tag{4.3.37}$$

and similarly as before we conclude that

$$V^kU^l = \mathrm{e}^{\frac{2\pi i}{n}lk}U^lV^k. \tag{4.3.38}$$

So the commutator between U and V is given by [7]

$$\begin{aligned} VUV^\dagger U^\dagger &= \mathrm{e}^{\frac{2\pi i}{n}}, \\ VUV^\dagger &= \mathrm{e}^{\frac{2\pi i}{n}}U, \\ V^kU^l(V^\dagger)^k(U^\dagger)^l &= \mathrm{e}^{\frac{2\pi i}{n}kl}, \\ V^kU^l(V^\dagger)^k &= \mathrm{e}^{\frac{2\pi i}{n}kl}U^l. \end{aligned} \tag{4.3.39}$$

Also we can write (4.3.37) and (4.3.38) in terms of $V^\dagger$, obtaining

$$\begin{aligned} V^\dagger U &= \mathrm{e}^{-\frac{2\pi i}{n}}UV^\dagger, \\ (V^\dagger)^kU^l &= \mathrm{e}^{-\frac{2\pi i}{n}lk}U^l(V^\dagger)^k. \end{aligned} \tag{4.3.40}$$

Finally, let us consider the algebra obtained from the representation of U and V on the $n \times n$ complex matrices linear space, and take the algebra commutators, which are

$$[V, U] = VU - UV = \left(\mathrm{e}^{\frac{2\pi i}{n}} - 1\right) UV \tag{4.3.41}$$

$$[V^\dagger, U] = V^\dagger U - UV^\dagger = \left(\mathrm{e}^{-\frac{2\pi i}{n}} - 1\right) UV^\dagger. \tag{4.3.42}$$

Now consider U and V separately: we can construct two discrete unitary groups whose elements are obtained by raising U and V to integer powers; by construction, both groups will have n distinct elements. Since U and V are unitary operators, they can be written as the complex exponential of two different Hermitian matrices. Nevertheless, we are not allowed to refer to these matrices as to group generators. In fact the two groups are discrete, and we cannot consider infinitesimal transformations: this means that we cannot derive the group commutator (4.3.37) from the commutator between two generators, as we have done instead for the continuous Weyl group.

The continuum limit of the results shown in this section, obtained by taking $n \to \infty$, yelds the continuous Weyl group, as Weyl shows in his book. We are interested in the discrete group, since in the next section we will use this formalism to replace the non unitary comparator of the Quantum Link Model with a unitary one.

4.4 Definition of a unitary comparator using the discrete Weyl group

Let us turn to a physical example and consider the situation in which the electric field and the vector potential are continuous; since they are defined on lattice links, the Hilbert space corresponding to each link must be infinite-dimensional. The two fields are represented respectively by two sets of Hermitian operators, $\{\hat{E}_{x,x+1}\}_x$ and $\{\hat{A}_{x,x+1}\}_x$, whose commutation relation is

$$[\hat{E}_{x,x+1}, \hat{A}_{x',x'+1}] = i\delta_{x,x'}. \tag{4.4.1}$$

Operators on different links commute with each other, since they act on different Hilbert spaces; instead, the electric field and the vector potential taken on the same link satisfy the Heisenberg's algebra product, and therefore their exponentials provide a representation of the Weyl group. We define two operators for each link: $\tilde{V}_{x,x+1}$ and its comparator $\tilde{U}_{x,x+1}$,

$$\tilde{U}_{x,x+1} = \mathrm{e}^{-i\hat{A}_{x,x+1}} \tag{4.4.2}$$

$$\tilde{V}_{x,x+1} = \mathrm{e}^{i\hat{E}_{x,x+1}}. \tag{4.4.3}$$

They belong to a continuous, two-parameter unitary group whose elements are obtained by raising $\tilde{U}_{x,x+1}$ and $\tilde{V}_{x,x+1}$ to real powers. The group commutator is

$$\tilde{V}^{\dagger}_{x,x+1}\tilde{U}^{\dagger}_{x,x+1}\tilde{V}_{x,x+1}\tilde{U}_{x,x+1} = \mathrm{e}^{i}, \tag{4.4.4}$$

as it follows by applying (4.2.9).

We now consider our model, in which the Hilbert space corresponding to each lattice link is finite-dimensional: henceforth the dimension will be n. The definition of a unitary comparator requires the implementation, on each link, of a n-dimensional representation of the discrete Weyl group: that is, $\tilde{U}_{x,x+1}$ and $\tilde{V}_{x,x+1}$, defined in (4.4.2) and (4.4.3), are replaced on each link with two operators which have the same features of U and V summarized in (4.3.40). Therefore we define on each link two new operators, $U_{x,x+1}$ and $V_{x,x+1}$:

$$U^{n}_{x,x+1} = \mathbb{1}, \tag{4.4.5}$$

$$V^{n}_{x,x+1} = \mathbb{1}, \tag{4.4.6}$$

$$V^{l}_{x,x+1}U^{k}_{x,x+1}\left(V^{\dagger}_{x,x+1}\right)^{l} = \mathrm{e}^{i\frac{2\pi}{n}kl}U^{k}_{x,x+1}, \tag{4.4.7}$$

$$V^{l}_{x,x+1}U^{k}_{x',x'+1} = U^{k}_{x',x'+1}V^{l}_{x,x+1}, \quad x \neq x'. \tag{4.4.8}$$

The operator $U_{x,x+1}$ is the new comparator for our model.

Note that l and k in (4.4.7), (4.4.8) are integers, and operators defined on different links commute with each other. Henceforth we will refer to operators of the discrete Weyl group as $U_{x,x+1}$ and $V_{x,x+1}$, while $\tilde{U}_{x,x+1}$ and $\tilde{V}_{x,x+1}$ will indicate the operators defined in (4.4.2), (4.4.3).

As in the previous section we chose as a basis the eigenvectors of V, now for each lattice link's Hilbert space we choose as a basis the eigenvectors of $V_{x,x+1}$, namely $\{|v_{k,\,x,x+1}\rangle\}_{1\leq k\leq n}$. We label the eigenvalues as suggested in (4.3.33). The action of $U_{x,x+1}$ and $V_{x,x+1}$ on $\{|v_{k,\,x,x+1}\rangle\}_{1\leq k\leq n}$ is

$$U_{x,x+1}|v_{k,\,x,x+1}\rangle = |v_{k+1,\,x,x+1}\rangle \tag{4.4.9}$$

$$U_{x,x+1}|v_{n,\,x,x+1}\rangle = |v_{1,\,x,x+1}\rangle \tag{4.4.10}$$

$$V_{x,x+1}|v_{k,\,x,x+1}\rangle = \mathrm{e}^{i\frac{2\pi}{n}k_{x,x+1}}|v_{k,\,x,x+1}\rangle, \quad -\frac{n}{2} \leq k_{x,x+1} \leq \frac{n}{2} - 1. \tag{4.4.11}$$

Let us examine from another point of view the passage from the continuous to the discrete Weyl group's operators, for an arbitrary lattice link: taken a real coefficient α, consider the element of the continuous Weyl group

$$\mathrm{e}^{i\alpha\hat{E}_{x,x+1}}, \tag{4.4.12}$$

whose spectrum is

$$\{\mathrm{e}^{i\alpha E_{x,x+1}}\}, \qquad E_{x,x+1} \in \mathbb{R}. \tag{4.4.13}$$

Instead, given $l \in \mathbb{Z}$, an arbitrary element of the discrete Weyl group is $V^{l}_{x,x+1}$, whose spectrum is

$$\{\mathrm{e}^{i\frac{2\pi}{n}lk_{x,x+1}}\}, \qquad k_{x,x+1} \in \mathbb{Z}. \tag{4.4.14}$$

Therefore the transition from the continuous to the discrete group leads to the following replacements of the group parameter and the electric field spectrum [23]

$$\alpha \to \sqrt{\frac{2\pi}{n}}\, l \qquad \alpha \in \mathbb{R},\ l \in \mathbb{Z} \tag{4.4.15}$$

$$E_{x,x+1} \to \sqrt{\frac{2\pi}{n}}\, k_{x,x+1}, \qquad E_{x,x+1} \in \mathbb{R},\ k_{x,x+1} \in \mathbb{Z}. \tag{4.4.16}$$

By construction, we consider as group parameter only the integer number, l in (4.4.14), and not its product by $\sqrt{2\pi/n}$. Similarly, henceforth we will refer to $k_{x,x+1}$ in (4.4.14) as to the electric field value correspondent to state $|v_{k,\,x,x+1}\rangle$ with eigenvalue $\mathrm{e}^{i\frac{2\pi}{n}k_{x,x+1}}$. We define on each link the following electric field operator:

$$\hat{k}_{x,x+1} = \begin{pmatrix} -\frac{n}{2} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{n}{2}+1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \frac{n}{2}-2 & 0 \\ 0 & 0 & 0 & 0 & \frac{n}{2}-1 \end{pmatrix}, \text{for even } n, \tag{4.4.17}$$

with a similar definition for odd n. Indeed, even if $k_{x,x+1}$ can take infinite values in $\mathbb{Z}$, the basis we are working with is that of $V_{x,x+1}$'s eigenvectors, and the Hilbert space is n-dimensional; also, due to the periodicity of function $\mathrm{e}^{i2\pi/n\,k_{x,x+1}}$, the electric field values differing by n generate the same eigenvalue of $V_{x,x+1}$, and therefore refer to the same basis vector. Hence we represent the electric field with a bounded and discrete operator, $\hat{k}_{x,x+1}$, whose complex exponential yields $V_{x,x+1}$:

$$\mathrm{e}^{i\frac{2\pi}{n}\hat{k}_{x,x+1}} = V_{x,x+1}. \tag{4.4.18}$$

With this choice the electric field $\hat{k}_{x,x+1}$ in our model has the same integer-valued spectrum of the electric field in the Quantum Link Model defined in (3.1.5). The comparators, in the Quantum Link Model and in our model, are those which distinguish the two situations. If we consider, in both models, electric field eigenvectors corresponding to the maximum and minimum eigenvalues, in the Quantum Link Model from (3.1.6) and (3.1.7) we have that

$$\begin{aligned} U^{\dagger}_{QLM\,x,x+1}|E_{QLM\,x,x+1} = -S\rangle_{QLM} = S^{+\dagger}_{x,x+1}|E_{QLM\,x,x+1} = -S\rangle_{QLM} = 0 \\ U_{QLM\,x,x+1}|E_{QLM\,x,x+1} = S\rangle_{QLM} = S^{+}_{x,x+1}|E_{QLM\,x,x+1} = S\rangle_{QLM} = 0, \end{aligned} \tag{4.4.19}$$

while in our model

$$U^{\dagger}_{x,x+1}|-n/2\rangle = |n/2-1\rangle, \quad U_{x,x+1}|n/2-1\rangle = |-n/2\rangle. \tag{4.4.20}$$

Relations (4.4.19) and (4.4.20) suggest the following graphical representation of the electric field spectrum in the different models: the continuous electric field

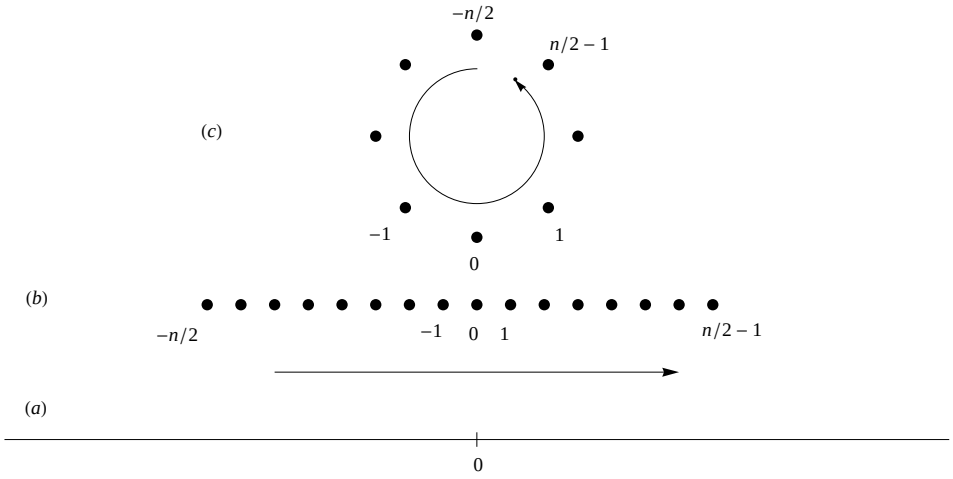

Figure 4.1: The spectra of: the continuous electric field (a), the QLM electric field (b), the electric field operator $\hat{k}_{x,x+1}$ (c) are displayed. The arrows show the direction of increasing electric field.

can be represented by a line, since it takes values in $\mathbb{R}$; the electric field in the Quantum Link Model can be represented with n points on a line, since it consists of a discrete and truncated version of the continuous field. Instead, the electric field values in our model can be represented as n points placed on a circle, at equal distance. Figure 4.1 displays this graphical representation of the different electric fields' spectra.

The definition of the unitary comparator using the discrete Weyl group will be useful to study the possibility to implement a gauge theory in our model: this is the topic of the next chapter.

Chapter 5

Implementation of a local $\mathbb{Z}_n$ symmetry on a lattice

In this chapter the implementation of a local $\mathbb{Z}_n$ symmetry on a lattice is presented. The passage from a continuous symmetry group to a discrete one requires a number of steps, which are analyzed in following sections.

First, we begin by studying a model without any fermion mass or electric field energy terms in the Hamiltonian. In this part, the new operators defined in the previous chapter will be used to define the Hamiltonian of our model and determine the form of a finite gauge transformation. Then, the consistence of this formalism will be checked; in particular Gauss' law will be examined to understand how it selects physical states.

Finally, we consider a dynamical term for the comparator. Its possible forms are studied together with the ground state of the theory.

5.1 Gauge transformations

The Hamiltonian we will refer to in this section is the following [2]:

$$\tilde{H} = -t \sum_x \psi_x^\dagger U_{x,x+1} \psi_{x+1} + \text{h.c.}, \tag{5.1.1}$$

in which x is an integer labeling lattice sites. The operators ψ_x are the fermion fields, and obey the anticommutators

$$\{\psi_x, \psi_{x'}^\dagger\} = \delta_{x,x'}, \quad \{\psi_x, \psi_{x'}\} = 0, \quad \{\psi_x^\dagger, \psi_{x'}^\dagger\} = 0. \tag{5.1.2}$$

Operator $U_{x,x+1}$ is the unitary comparator defined in the previous chapter by relations (4.4.7) and (4.4.8). Together with $U_{x,x+1}$, the operator $V_{x,x+1}$ is defined by the relations (4.4.7) and (4.4.8) too, and they satisfy

$$V_{x,x+1}^l U_{x,x+1}^k \left(V_{x,x+1}^\dagger\right)^l = \mathrm{e}^{i\frac{2\pi}{n}kl} U_{x,x+1}^k, \tag{5.1.3}$$

$$\left(V_{x,x+1}^\dagger\right)^l U_{x,x+1}^k V_{x,x+1}^l = \mathrm{e}^{-i\frac{2\pi}{n}kl} U_{x,x+1}^k, \tag{5.1.4}$$

while operators on different links commute. The Hilbert space for our model is the tensor product of the Hilbert spaces relative to each link and each site; the basis we choose is

$$\left\{ \prod_x |n_x^F\rangle \otimes \prod_x |v_{x,x+1}\rangle \right\}, \tag{5.1.5}$$

in which $|n_x^F\rangle$ are the eigenvectors of the fermion number operators $\psi_x^\dagger \psi_x$ and $|v_{x,x+1}\rangle$ are the eigenvectors of $V_{x,x+1}$. By recalling the definition of $\hat{k}_{x,x+1}$ given in (4.4.17) and relation (4.4.18), we identify each eigenvector of $V_{x,x+1}$ with an eigenvector of the electric field operator $\hat{k}_{x,x+1}$.

The Hamiltonian (5.1.1) is symmetric under these local gauge transformations [7]:

$$\psi_x \to \mathrm{e}^{i\frac{2\pi}{n}\alpha_x}\, \psi_x \tag{5.1.6}$$

$$U_{x,x+1} \to \mathrm{e}^{i\frac{2\pi}{n}\alpha_x}\, U_{x,x+1}\, \mathrm{e}^{-i\frac{2\pi}{n}\alpha_{x+1}}, \tag{5.1.7}$$

in which α_x is a real function defined on the lattice sites. If the function α_x takes only integer values we can implement transformations (5.1.6) and (5.1.7) as follows:

$$\psi_x \to \mathrm{e}^{i\frac{2\pi}{n}\alpha_x \psi_x^\dagger \psi_x}\, \psi_x\, \mathrm{e}^{-i\frac{2\pi}{n}\alpha_x \psi_x^\dagger \psi_x}, \tag{5.1.8}$$

$$U_{x,x+1} \to (V_{x,x+1}^\dagger)^{\alpha_{x+1}} (V_{x,x+1})^{\alpha_x}\, U_{x,x+1}\, (V_{x,x+1}^\dagger)^{\alpha_x} (V_{x,x+1})^{\alpha_{x+1}}. \tag{5.1.9}$$

The request to take integer-valued functions α_x is essential to write (5.1.9), since it is an immediate application of commutator (5.1.3); transformation (5.1.8) follows, instead, by observing that since

$$\mathrm{e}^{i\alpha_x \psi_x^\dagger \psi_x 2\pi/n} = 1 + \left(\mathrm{e}^{i\alpha_x 2\pi/n} - 1\right) \psi_x^\dagger \psi_x, \tag{5.1.10}$$

we have

$$\mathrm{e}^{i\frac{2\pi}{n}\alpha_x \psi_x^\dagger \psi_x} \psi_x = \psi_x, \tag{5.1.11}$$

$$\mathrm{e}^{i\frac{2\pi}{n}\alpha_x \psi_x^\dagger \psi_x} \psi_x^\dagger = \mathrm{e}^{i\frac{2\pi}{n}\alpha_x} \psi_x^\dagger, \tag{5.1.12}$$

$$\psi_x \mathrm{e}^{i\frac{2\pi}{n}\alpha_x \psi_x^\dagger \psi_x} = \mathrm{e}^{i\frac{2\pi}{n}\alpha_x} \psi_x, \tag{5.1.13}$$

$$\psi_x^\dagger \mathrm{e}^{i\frac{2\pi}{n}\alpha_x \psi_x^\dagger \psi_x} = \psi_x^\dagger. \tag{5.1.14}$$

On-site and on-link transformations defined in (5.1.8) and (5.1.9) can be put together to implement a local transformation acting on the whole lattice Hamiltonian. In order to do this we define

$$T[\alpha_x] = \prod_x T_x(\alpha_x) = \prod_x e^{\frac{2\pi i}{n}\alpha_x \psi_x^\dagger \psi_x} e^{\frac{2\pi i}{n}\alpha_x \frac{1}{2}[(-1)^x - 1]} \otimes (V_{x,x+1}^\dagger)^{\alpha_x} (V_{x-1,x})^{\alpha_x}. \tag{5.1.15}$$

The local transformation of the Hamiltonian reads

$$\tilde{H} \to T^\dagger[\alpha_x]\, \tilde{H}\, T[\alpha_x]. \tag{5.1.16}$$

The constraint of considering integer-valued functions α_x leads us from local transformations of $U(1)$ to local transformations of $\mathbb{Z}_n$: this group is abstractly constructed by raising a group element u, called generator, to integer powers, and by imposing the condition $u^n = \mathbb{1}$. So the group consists of n different elements, $\{\mathbb{1}, u, u^2, u^3, \ldots, u^{n-1}\}$. The most natural representation of this group is that of finite rotations by angles which differ by $2\pi/n$. In our case the generators of the group form a set of operators for each lattice site, since we are interested in local transformations, and they take the form

$$u_x = e^{\frac{2\pi i}{n}\psi_x^\dagger\psi_x} e^{\frac{2\pi i}{n}\frac{1}{2}[(-1)^x - 1]} \otimes (V_{x,x+1}^\dagger)(V_{x-1,x}), \qquad (u_x)^n = \mathbb{1}. \tag{5.1.17}$$

In the following we will explicitly verify the local gauge invariance of the system under transformations (5.1.16), by computing the commutator between the Hamiltonian and a local transformation $T[\alpha_x]$. In particular, the local gauge invariance is guaranteed by the condition

$$[T_x(\alpha_x), \tilde{H}] = 0. \tag{5.1.18}$$

Written explicitly, the commutator reads

$$\begin{aligned}[T_x(\alpha_x), \tilde{H}] = &- t\ T_x(\alpha_x) \sum_y \psi_y^\dagger U_{y,y+1}\psi_{y+1} + t \sum_y \psi_y^\dagger U_{y,y+1}\psi_{y+1} T_x(\alpha_x) \\ &- t\ T_x(\alpha_x) \sum_y \psi_y U_{y,y+1}^\dagger \psi_{y+1}^\dagger + t \sum_y \psi_y U_{y,y+1}^\dagger \psi_{y+1}^\dagger T_x(\alpha_x).\end{aligned} \tag{5.1.19}$$

Considering, as an example, the first of the four terms in the right hand side we have that

$$\begin{aligned}&T_x(\alpha_x) \sum_y \psi_y^\dagger U_{y,y+1}\psi_{y+1} \\ = &\underbrace{\sum_{y\neq x,x-1} \psi_y^\dagger U_{y,y+1}\psi_{y+1} T_x}_{(a)} \\ &+ \underbrace{\mathrm{e}^{\frac{2\pi i}{n}\alpha_x[\psi_x^\dagger\psi_x + \frac{1}{2}[(-1)^x - 1]]} \otimes (V_{x,x+1}^\dagger)^{\alpha_x}(V_{x-1,x})^{\alpha_x}\psi_x^\dagger U_{x,x+1}\psi_{x+1}}_{(b)} \\ &+ \underbrace{\mathrm{e}^{\frac{2\pi i}{n}\alpha_x[\psi_x^\dagger\psi_x + \frac{1}{2}[(-1)^x - 1]]} \otimes (V_{x,x+1}^\dagger)^{\alpha_x}(V_{x-1,x})^{\alpha_x}\psi_{x-1}^\dagger U_{x-1,x}\psi_x}_{(c)}.\end{aligned} \tag{5.1.20}$$

The term (a) is justified by observing that all operators in the sum act on sites and links different from those present in $T_x(\alpha_x)$; also, V and U defined on different links commute with each other. Finally, we use (5.1.10) and observe that, for $y \neq x$ and $y \neq x-1$,

$$\begin{aligned}\psi_x^\dagger\psi_x\psi_y^\dagger\psi_{y+1} = &- \psi_x^\dagger\psi_y^\dagger\psi_x\psi_{y+1} = -\psi_y^\dagger\psi_x^\dagger\psi_{y+1}\psi_x \\ &= \psi_y^\dagger\psi_{y+1}\psi_x^\dagger\psi_x,\end{aligned} \tag{5.1.21}$$

that is, fermion number operators commute with fermion fields defined on different sites. Now let us compute the term (b), without considering constant factors, since they do not influence commutators:

$$\begin{aligned}
&\mathrm{e}^{\frac{2\pi i}{n}\alpha_x\psi_x^\dagger\psi_x}\otimes(V_{x,x+1}^\dagger)^{\alpha_x}(V_{x-1,x})^{\alpha_x}\psi_x^\dagger U_{x,x+1}\psi_{x+1}\\
=&\mathrm{e}^{\frac{2\pi i}{n}\alpha_x}\psi_x^\dagger\psi_{x+1}(V_{x,x+1}^\dagger)^{\alpha_x}U_{x,x+1}(V_{x-1,x})^{\alpha_x}\\
=&\psi_x^\dagger\psi_{x+1}\mathrm{e}^{\frac{2\pi i}{n}\alpha_x}\mathrm{e}^{-\frac{2\pi i}{n}\alpha_x}U_{x,x+1}(V_{x,x+1}^\dagger)^{\alpha_x}(V_{x-1,x})^{\alpha_x}\\
=&\psi_x^\dagger\psi_{x+1}U_{x,x+1}(V_{x,x+1}^\dagger)^{\alpha_x}(V_{x-1,x})^{\alpha_x}\\
=&\psi_x^\dagger\mathrm{e}^{\frac{2\pi i}{n}\alpha_x\psi_x^\dagger\psi_x}\psi_{x+1}U_{x,x+1}(V_{x,x+1}^\dagger)^{\alpha_x}(V_{x-1,x})^{\alpha_x}\\
=&\psi_x^\dagger U_{x,x+1}\psi_{x+1}\mathrm{e}^{\frac{2\pi i}{n}\alpha_x\psi_x^\dagger\psi_x}(V_{x,x+1}^\dagger)^{\alpha_x}(V_{x-1,x})^{\alpha_x}.
\end{aligned}\tag{5.1.22}$$

Relations (5.1.12) and (5.1.14) have been used, together with (5.1.3); the term (c) similarly leads to the same result. Inserting these results in the right hand side of (5.1.19) one finds that the first term cancels out with the second, and similarly the two other terms vanish.

With the above calculation we have shown the local gauge invariance of the system under transformations (5.1.16); therefore, it is possible to implement a gauge theory on a lattice with a n-dimensional Hilbert space of states on each link. The local symmetry we have implemented in the Hamiltonian (5.1.1) is that under transformations of $\mathbb{Z}_n$. To do this, we have used the discrete Weyl group to define the unitary comparator $U_{x,x+1}$ in the Hamiltonian, and the operator $V_{x,x+1}$ in $T[\alpha_x]$.

Now, as a comment, let us recall briefly the implementation of an Abelian gauge theory on a lattice with a continuous electric field $\hat{E}_{x,x+1}$ and a vector potential $\hat{A}_{x,x+1}$, defined through the commutator (4.4.1), in order to stress some differences with the discrete case. The Hamiltonian is formally the same of (5.1.1),

$$H_{cont}=-t\sum_x\psi_x^\dagger\tilde{U}_{x,x+1}\psi_{x+1}+\text{h.c.},\tag{5.1.23}$$

but $\tilde{U}_{x,x+1}$ is the comparator defined in (4.4.3); from the commutator (4.4.1) between $\hat{E}_{x,x+1}$ and $\hat{A}_{x,x+1}$ it follows that

$$[\hat{E}_{x,x+1},\tilde{U}_{x',x'+1}]=\delta_{x,x'}\tilde{U}_{x,x+1}.\tag{5.1.24}$$

Now we can define a finite local transformation, by taking the operator

$$\begin{aligned}
\tilde{T}[\alpha_x]&=\prod_x\tilde{T}_x=\prod_x\mathrm{e}^{i\alpha_xG_x}\\
&=\prod_x\mathrm{e}^{i\alpha_x[\psi_x^\dagger\psi_x+\frac{1}{2}[(-1)^x-1]-(\hat{E}_{x,x+1}-\hat{E}_{x-1,x})]},
\end{aligned}\tag{5.1.25}$$

where α_x is a real function defined on lattice sites and G_x are the generators for infinitesimal transformations. A finite local symmetry transformation, given by

$$H_{cont}\to\tilde{T}^\dagger[\alpha_x]\,H_{cont}\,\tilde{T}[\alpha_x],\tag{5.1.26}$$

leaves the Hamiltonian unchanged: this result is a consequence of $[G_x, H_{cont}] = 0$, which holds thanks to (5.1.24).

The difference between $T[\alpha_x]$ defined in (5.1.15) and $\tilde{T}[\alpha_x]$ defined in (5.1.25) is in the operators which transform the comparators in the Hamiltonian $\tilde{H}$ and H_{cont}, respectively: they are $V_{x,x+1}$ in the first case and $\mathrm{e}^{i\hat{E}_{x,x+1}}$ in the second one. In the first case the gauge invariance is guaranteed by the group commutator (5.1.3), in the second by the algebra commutator (4.4.1).

In conclusion, if we consider continuous electric field and vector potential we can implement a local symmetry with respect to a continuous group of transformations, $U(1)$: the gauge invariance can be verified by studying the commutator $[\tilde{T}_x, H_{cont}]$ or $[G_x, H_{cont}]$ as well. Instead, if one deals with a discrete electric field, the local symmetry group is $\mathbb{Z}_n$, and the gauge invariance of the Hamiltonian must be proved under finite transformations, as done in (5.1.19).

5.2 Physical states

We briefly recall how physical states are defined in the model in which the electric field and the vector potential are continuous. We will then study physical states in our system. From the definition of generators G_x in (5.1.25) it follows that

$$G_x = \psi_x^\dagger \psi_x + \frac{1}{2}[(-1)^x - 1] - (\hat{E}_{x,x+1} - \hat{E}_{x-1,x}). \tag{5.2.1}$$

We take as a basis of the Hilbert space $\left\{ \prod_x |n_x^F\rangle \otimes \prod_x |e_{x,x+1}\rangle \right\}$, in which $\{|n_x^F\rangle\}$ are the eigenvectors of the fermion number operator on site x and $\{|e_{x,x+1}\rangle\}$ are the electric field's eigenvectors on link $x, x+1$. The subspace of physical states is characterized by basis vectors $|\psi\rangle$ which obey Gauss' law:

$$G_x|\psi\rangle = 0, \qquad \forall x. \tag{5.2.2}$$

Note that Gauss' law imposes the local gauge invariance of physical states, since according to (5.1.25) it follows that

$$\tilde{T}_x|\psi\rangle = |\psi\rangle, \tag{5.2.3}$$

and leads to the following local condition on the fermion number n_x^F and on the electric field eigenvalues $E_{x,x+1}$, $E_{x-1,x}$:

$$n_x^F + \frac{1}{2}[(-1)^x - 1] - (E_{x,x+1} - E_{x-1,x}) = 0, \qquad \forall x. \tag{5.2.4}$$

Now we consider our model, in which the Hilbert space related to each link is n-dimensional: local symmetry transformations (5.1.16) have been defined through the operator:

$$T = \prod_x T_x = \prod_x e^{\frac{2\pi i}{n}\alpha_x \psi_x^\dagger \psi_x} e^{\frac{2\pi i}{n}\alpha_x \frac{1}{2}[(-1)^x - 1]} \otimes (V_{x,x+1}^\dagger)^{\alpha_x} (V_{x-1,x})^{\alpha_x}. \tag{5.2.5}$$

Recall that the numerical factor related to the parity of the site, $\frac{1}{2}\left[(-1)^x - 1\right]$, is due to the use of staggered fermions. The basis of the Hilbert space $\mathcal{H}$ we are working with is $\left\{ \prod_x |n_x^F\rangle \otimes \prod_x |v_{x,x+1}\rangle \right\}$: as defined in (5.1.5), $|n_x^F\rangle$ and $|v_{x,x+1}\rangle$ are the eigenvectors of $\psi_x^\dagger \psi_x$ and $V_{x,x+1}$, respectively.

Imposing Gauss' law leads to a local gauge invariance condition for physical states: a physical state $|\Phi\rangle$ must obey the relation

$$T_x|\Phi\rangle = |\Phi\rangle \qquad \forall x. \tag{5.2.6}$$

Let us take a basis vector $|\phi\rangle$: by taking the action of T_x on $|\phi\rangle$, condition (5.2.6) reads

$$\begin{aligned} T_x|\phi\rangle &= \mathrm{e}^{\frac{2\pi i}{n}\alpha_x \psi_x^\dagger \psi_x} e^{\frac{2\pi i}{n}\alpha_x \frac{1}{2}[(-1)^x - 1]} (V_{x,x+1}^\dagger)^{\alpha_x} (V_{x-1,x})^{\alpha_x} |\phi\rangle \\ &= \mathrm{e}^{\frac{2\pi i}{n}\alpha_x [n_x^F + \frac{1}{2}[(-1)^x - 1]]} (v_{x,x+1}^*)^{\alpha_x} (v_{x-1,x})^{\alpha_x} |\phi\rangle \\ &= |\phi\rangle. \end{aligned} \tag{5.2.7}$$

We need condition (5.2.6) to be independent from the choice of the function α_x; therefore, states $|\phi\rangle$ must be such that for all x the eigenvalues of the operators $V_{x-1,x}$, $V_{x,x+1}^\dagger$ and $\psi_x^\dagger \psi_x$ obey the relation

$$\mathrm{e}^{\frac{2\pi i}{n}[n_x^F + \frac{1}{2}[(-1)^x - 1]]} v_{x,x+1}^* v_{x-1,x} = 1. \tag{5.2.8}$$

By imposing condition (5.2.8) for each site x we select those basis vectors whose linear combinations generate physical states of $\mathcal{H}$: these states obey to the same condition (5.2.6) and are locally gauge invariant.

Now, let us recall the definition (4.4.17) of the electric field operator $\hat{k}_{x,x+1}$:

$$\hat{k}_{x,x+1} = \begin{pmatrix} -\frac{n}{2} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{n}{2}+1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \frac{n}{2}-2 & 0 \\ 0 & 0 & 0 & 0 & \frac{n}{2}-1 \end{pmatrix}, \text{for even } n.$$

Since $v_{x,x+1} = \mathrm{e}^{\frac{2\pi i}{n} k_{x,x+1}}$, we can express (5.2.8) in an equivalent way which involves the electric field eigenvalues $k_{x,x+1}$, by making use of congruence modulo n. Two integer numbers a and b are congruent modulo n, namely

$$a \equiv b \mod n, \tag{5.2.9}$$

if their difference is an integer multiple of n. With this definition, Gauss' law (5.2.8) can be rewritten as following:

$$\begin{aligned} n_x^F + \frac{1}{2}[(-1)^x - 1] &= \Delta_x, \\ \Delta_x &\equiv k_{x,x+1} - k_{x-1,x} \mod n. \end{aligned} \tag{5.2.10}$$

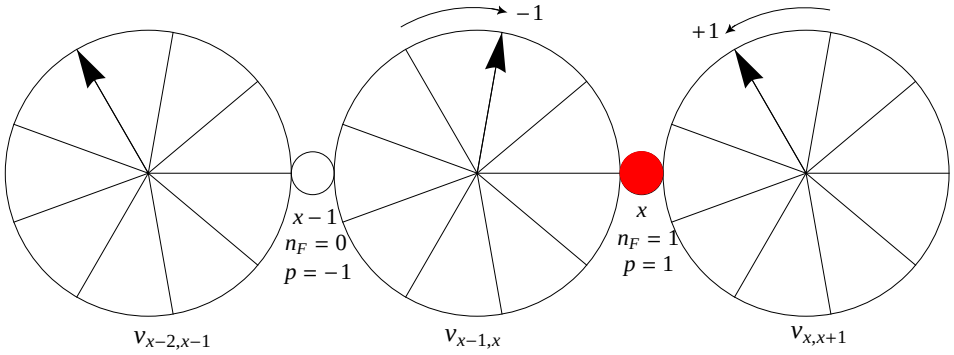

Figure 5.1: A particular of a physical (gauge invariant) state: $p = (-1)^x$ is the parity of the site, and red dots indicate the presence of a positively charged fermion in site. In the circles the phasors corresponding to eigenvalues of $V_{x,x+1}$ are displayed, while the external arrows stress the increasing or decreasing of the electric field on links from left to right.

Congruence modulo n is an equivalence relation which divides integer numbers in n equivalence classes, $\{[0], [1], \ldots, [n-1]\}$. These classes can be labelled with an arbitrary set of n integer numbers, that are the possible values for $k_{x,x+1}$: according to definition (4.4.17) of $\hat{k}_{x,x+1}$, we choose $\{-n/2, \ldots, n/2 - 1\}$ with even n, $\{-(n-1)/2, \ldots, (n-1)/2\}$ if n is odd. In particular, for even n it follows that:

$$\begin{aligned} \left(\frac{n}{2} - 1\right) + 1 = \frac{n}{2} \equiv -\frac{n}{2} \mod n \\ \left(\frac{n}{2} - 1\right) - \left(-\frac{n}{2}\right) = n - 1 \equiv -1 \mod n, \end{aligned} \tag{5.2.11}$$

and analogue expression ca be written for odd n.

A number of comments are needed. First, Δ_x assume only three values, which are $0, \pm 1$. This form of Gauss' law differs from that obtained with the Quantum Link Model for the term $k_{x,x+1} - k_{x-1,x}$, which replaces $(E_{x,x+1} - E_{x-1,x})$ in (5.2.4), and for the use of congruence modulo n.

From relation (5.2.10) it follows that Gauss' law involves the fermion number on a certain site, and the electric field values on the adjacent links. By assuming that fermions on sites brings a positive electric charge, given a site x three different situations are possible:

- $\Delta_x = 1$: this is the case when parity is positive and $n^F = 1$. In this case

$$k_{x,x+1} = k_{x-1,x} + 1 \mod n;$$

- $\Delta_x = 0$: this is the case when parity is positive and $n^F = 0$, or when parity is negative and $n^F = 1$. In this case the electric field remains constant,

$$k_{x,x+1} = k_{x-1,x};$$

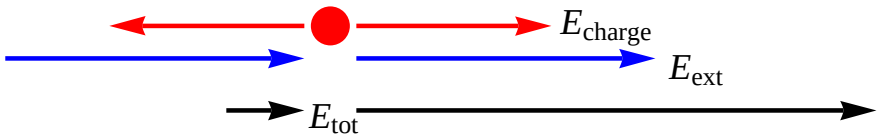

Figure 5.2: The red dot stands for the positive charge, and the electric field it generates, E_{charge}, is displayed in red. The external electric field E_{ext}, supposed uniform, is represented in blue. The total electric field E_{tot}, on the left and on the right of the charge, is given by the sum of them.

- $\Delta_x = -1$: this is the case when parity is negative and $n^F = 0$. In this case

$$k_{x,x+1} = k_{x-1,x} - 1 \mod n.$$

These three cases show how, in our model, the presence of a positive charge on a site, combined with the parity of the site itself, modify the electric field. Pictorially, we can visualize the Gauss' law with a picture. Let us imagine to work with a positive charge and an external electric field: the total electric field will be given by the sum of the external electric field and the electric field generated by the positive charge itself: with reference to Figure 5.2, on the right of the charge the field will be increased, on the left it will be decreased.

We now focus on two particular physical states, constructed according to condition (5.2.10):

- $\Delta_x = 1$, $k_{x-1,x} = \frac{n}{2} - 1$: it follows that

$$k_{x,x+1} = -\frac{n}{2}$$

- $\Delta_x = -1$, $k_{x-1,x} = -\frac{n}{2}$: we have

$$k_{x,x+1} = \frac{n}{2} - 1.$$

For these states we have that $|k_{x,x+1} - k_{x-1,x}| = n-1$, and according to condition (5.2.10) such states are physical; in the Quantum Link Model, instead, the electric field difference between adjacent links can be only $0, \pm 1$. This new condition is quite counterintuitive since a unit charge increases or decreases the electric field of an amount of $n - 1$.

In order to understand this difference, we represent the eigenvalues of $V_{x,x+1}$ with complex phasors, one for each link. The eigenvalue $v_{x,x+1} = 1$ is represented by a phasor aligned with the positive horizontal axis. The angle of each phasor is the electric field eigenvalue, according to the definition of the electric field operator $\hat{k}_{x,x+1}$ and its relation with $V_{x,x+1}$ expressed by (4.4.18):

$$V_{x,x+1} = \mathrm{e}^{i\frac{2\pi}{n}\hat{k}_{x,x+1}}.$$

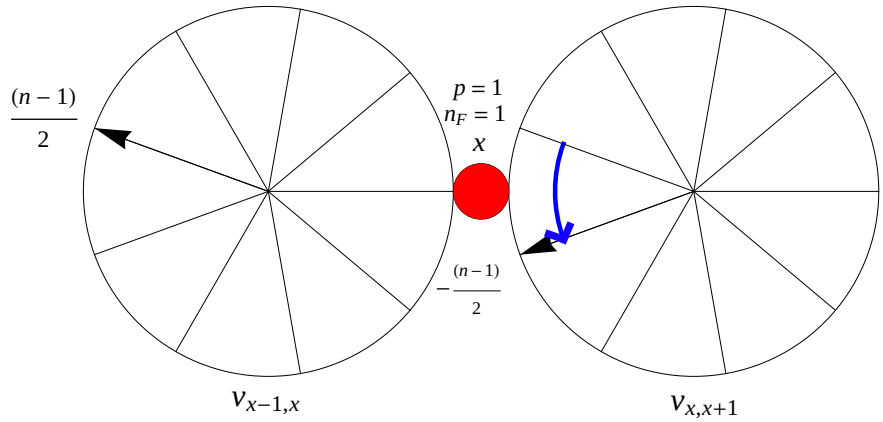

Figure 5.3: Two adjacent links with maximum and minimum eigenvalues of the electric field operator $\hat{k}_{x,x+1}$. Recall that p is the parity of the site.

As the phasor rotates counterclockwise the electric field increases, while a clockwise rotation decreases it. This representation emphasizes that Gauss' law formulated in (5.2.10) equates Δ_x with the angle difference, expressed in unit of $\frac{2\pi}{n}$, between two adjacent link's phasors.

In Figure 5.1 a detail of an arbitrary physical basis vector is represented, while in Figure 5.3 we have constructed a state which is physical in our model, but not in the Quantum Link Model, as discussed above: note that the phasor rotates by an angle of $2\pi/n$ from left to right, but the correspondent eigenvalue of the electric field passes from $k_{x-1,x} = (n-1)/2$ to $k_{x,x+1} = -(n-1)/2$.

We now will focus on another aspect, related to allowed transitions between physical states. Henceforth n will be taken odd for definiteness, but even n could be considered as well, yielding the same results. Let us consider, in a gauge invariant basis vector, link $x, x+1$ and its adjacent sites. Let us consider the state $|\phi\rangle$ containing the three vectors

$$|n_x^F = 1\rangle,\ |v_{x,x+1} = \mathrm{e}^{-\frac{2\pi i}{n}\frac{(n-1)}{2}}\rangle,\ |n_{x+1}^F = 0\rangle, \tag{5.2.12}$$

in which the electric field value in $x, x+1$ takes its minimum value, since $k_{x,x+1} = -\frac{(n-1)}{2}$. The allowed transitions of this state can be studied by taking

$$\tilde{H}|\phi\rangle = \left(-t\sum_y \psi_y^\dagger U_{y,y+1}\psi_{y+1} + \text{h.c.}\right)|\phi\rangle, \tag{5.2.13}$$

but we concentrate only on a single couple of terms in the Hamiltonian, namely

$$\left(-t\psi_x^\dagger U_{x,x+1}\psi_{x+1} + \text{h.c.}\right)|\phi\rangle. \tag{5.2.14}$$

In conclusion, we are focusing on the following process involving the vector

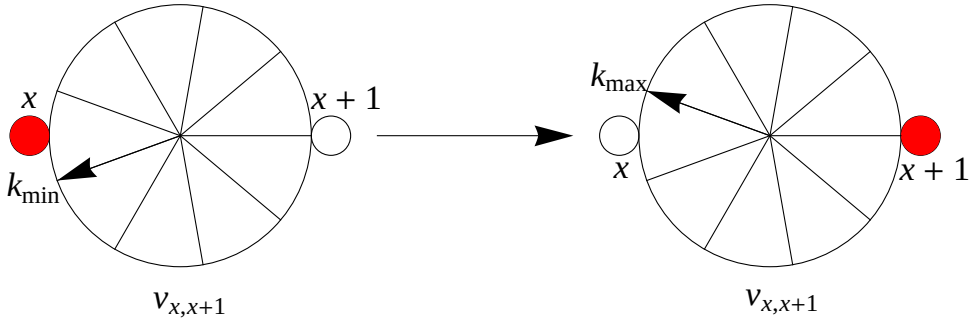

Figure 5.4: Evolution of a link and adjacent sites according to $\psi_x U^\dagger_{x,x+1}\psi^\dagger_{x+1}$.

(5.2.12) and the sum of two operators:

$$\begin{aligned}
&\left(-t\psi^\dagger_x U_{x,x+1}\psi_{x+1} + \text{h.c.}\right)|n^F_x = 1\rangle|v_{x,x+1} = \mathrm{e}^{-\frac{2\pi i}{n}\frac{(n-1)}{2}}\rangle|n^F_{x+1} = 0\rangle \\
&= -t\psi_x U^\dagger_{x,x+1}\psi^\dagger_{x+1}|n^F_x = 1\rangle|v_{x,x+1} = \mathrm{e}^{-\frac{2\pi i}{n}\frac{(n-1)}{2}}\rangle|n^F_{x+1} = 0\rangle \\
&= |n^F_x = 0\rangle|v_{x,x+1} = \mathrm{e}^{\frac{2\pi i}{n}\frac{(n-1)}{2}}\rangle|n^F_{x+1} = 1\rangle. \qquad (5.2.15)
\end{aligned}$$

One of the two operators has transformed the vector, yielding to the final result. The other operator annihilated it, due to the action of the fermion number operators, since $\psi^\dagger_x|n^F_x = 1\rangle = \psi_{x+1}|n^F_{x+1} = 0\rangle = 0$. If we were working in the Quantum Link model, also the second operator would have annihilated the state, due to the action of the non unitary comparator $U^\dagger_{QLM\,x,x+1}$. Indeed, $U^\dagger_{QLM\,x,x+1}|E_{x,x+1} = -\frac{(n-1)}{2}\rangle_{QLM} = 0$. In our model, instead, the unitary operator $U^\dagger_{x,x+1}$ rotates clockwise the phasor correspondent to $v_{x,x+1}$, by an angle of $\frac{2\pi}{n}$. It follows as a consequence that the state ϕ defined in (5.2.12) evolves as shown in (5.2.15), and the electric field passes from the value $k_{x,x+1} = -\frac{(n-1)}{2}$ to $k'_{x,x+1} = \frac{(n-1)}{2}$. The transition we have just discussed is shown in Figure 5.4.

In the same way, we can consider the symmetric case, in which the electric field is initially $k_{x,x+1} = \frac{n-1}{2}$ and the action of $U_{x,x+1}$ rotates counterclockise the phasor correspondent to $v_{x,x+1}$: the electric field value is then $k'_{x,x+1} = -\frac{n-1}{2}$.

5.3 Electric field energy and vacuum state

In this section we discuss the term that must be added to the Hamiltonian $\tilde{H}$ to implement the dynamics of the comparator and that replaces the energy of the electric field. We start from the observation that in the continuous field theory this term is given by

$$H_{dyn} = \frac{g^2}{2}\sum_x \hat{E}^2_{x,x+1}, \qquad (5.3.1)$$

where $\hat{E}_{x,x+1}$ is the electric field operator defined in (4.4.1). This is an unbounded, positive operator which reaches its minimum value when the electric

field is zero on each link. It follows that the vacuum state for the electric field on the lattice is the state with no electric field on each link.

In our model this term of the Hamiltonian must be defined as a function of the operators $V_{x,x+1}$ and $V^\dagger_{x,x+1}$ which in the continuum limit tends to the energy term (5.3.1). There are many such operators: in the following a possible choice will be discussed, stressing good and less appealing aspects. Let us define the term:

$$\begin{aligned}\tilde{H}_{dyn} &= \frac{g^2}{2}\sum_x \frac{(V_{x,x+1}-\mathbb{1})(V^\dagger_{x,x+1}-\mathbb{1})}{(2\pi/n)^2} \\ &= \frac{g^2}{2}\sum_x C_{x,x+1},\end{aligned} \tag{5.3.2}$$

in which the operators $C_{x,x+1}$ are defined *ad hoc* on each link. The operators $C_{x,x+1}$ satisfy a number of properties, thanks to which $\tilde{H}_{dyn}$ is a good candidate to replace the electric field energy term. First of all, for each link $C_{x,x+1}$ is Hermitian, therefore it is a good observable. Second, it is a gauge invariant quantity: since all the operators $V_{x,x+1}$ and $V^\dagger_{x,x+1}$ commute each other, it follows immediately that

$$\tilde{H}_{dyn} \to T^\dagger[\alpha_x]\,\tilde{H}_{dyn}\,T[\alpha_x] = \tilde{H}_{dyn}. \tag{5.3.3}$$

Now let us examine $\tilde{H}_{dyn}$ in the continuum limit: for $n \to +\infty$ the operator $C_{x,x+1}$ tends to $\hat{k}^2_{x,x+1}$. By considering the spectrum of $C_{x,x+1}$ we have

$$\begin{aligned}\frac{(V_{x,x+1}-\mathbb{1})(V^\dagger_{x,x+1}-\mathbb{1})}{(2\pi/n)^2}|v_{x,x+1}\rangle &= \frac{(2-V_{x,x+1}-V^\dagger_{x,x+1})}{(2\pi/n)^2}|v_{x,x+1}\rangle \\ &= \frac{2(1-\cos(2\pi k_{x,x+1}/n))}{(2\pi/n)^2}|v_{x,x+1}\rangle \\ &= \frac{4\sin^2(\pi k_{x,x+1}/n)}{(2\pi/n)^2}|v_{x,x+1}\rangle.\end{aligned} \tag{5.3.4}$$

By taking the limit

$$\lim_{n\to+\infty}\frac{4\sin^2(\pi k_{x,x+1}/n)}{(2\pi/n)^2} = k^2_{x,x+1}, \tag{5.3.5}$$

we obtain that $C_{x,x+1} \to \hat{k}^2_{x,x+1}$, indeed. Also, the operator $\hat{k}_{x,x+1}$, multiplied by the coefficient $\sqrt{2\pi/n}$, in the continuum limit tends to the continuous electric field: this emerges by recalling (4.4.16). Therefore we conclude that $\tilde{H}_{dyn}$ in the continuum limit tends to H_{dyn} defined in (5.3.1).

The spectrum of $C_{x,x+1}$ is bounded and discrete, since it is given by the product of unitary operators. Its minimum eigenvalue is zero, and corresponds to $k_{x,x+1} = 0$, or $v_{x,x+1} = 1$.

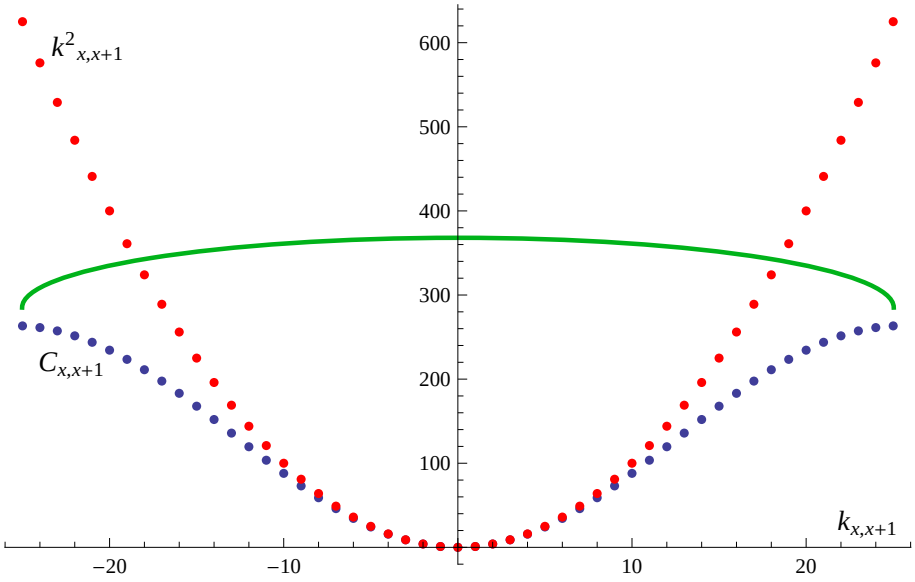

Figure 5.5: Spectra of $k^2_{x,x+1}$ (red) and $C_{x,x+1}$ (blue) compared.

Therefore the vacuum state in our model, as long the links are concerned, is the state in which for each link $v_{x,x+1} = 1$. In Figure 5.5 the spectra of the operators $C_{x,x+1}$ and $k^2_{x,x+1}$ are displayed. The green line shows the transition described in Figure 5.4 in which $k_{x,x+1}$ passes from its minimum to its maximum value. The two spectra, that of $C_{x,x+1}$ and that of $k^2_{x,x+1}$ both reach their maximum values at the edges of the interval. The spectrum of $k^2_{x,x+1}$ is a discrete and truncated version of the spectrum of the operator (5.3.1), but it keeps the original parabolic shape. In the case of $C_{x,x+1}$, instead, the spectrum is deformed at the edges of the interval, so only for low energy states $\tilde{H}_{dyn}$ approximates the electric field energy: the approximation gets better as n increases.

As a comment, we define an operator which in the continuum limit tends to $\hat{k}^2_{x,x+1}$, too, but which is not a good dynamical term for the comparator. Let us start by defining the Hermitian operator

$$S_{x,x+1} = \frac{V_{x,x+1} - V^\dagger_{x,x+1}}{2i(2\pi/n)}. \tag{5.3.6}$$

The operator $S^2_{x,x+1}$ in the continuum limit tends to $\hat{k}^2_{x,x+1}$: its spectrum can be derived as follows,

$$\begin{aligned} S^2_{x,x+1}|v_{x,x+1}\rangle &= \left[\frac{V - V^\dagger}{2i(2\pi/n)}\right]^2 |v_{x,x+1}\rangle \\ &= \frac{\sin^2(2\pi k_{x,x+1}/n)}{(2\pi/n)^2}|v_{x,x+1}\rangle, \end{aligned} \tag{5.3.7}$$

and since

$$\lim_{n\to\infty} \frac{\sin^2(2\pi k_{x,x+1}/n)}{(2\pi/n)^2} = k^2_{x,x+1} \tag{5.3.8}$$

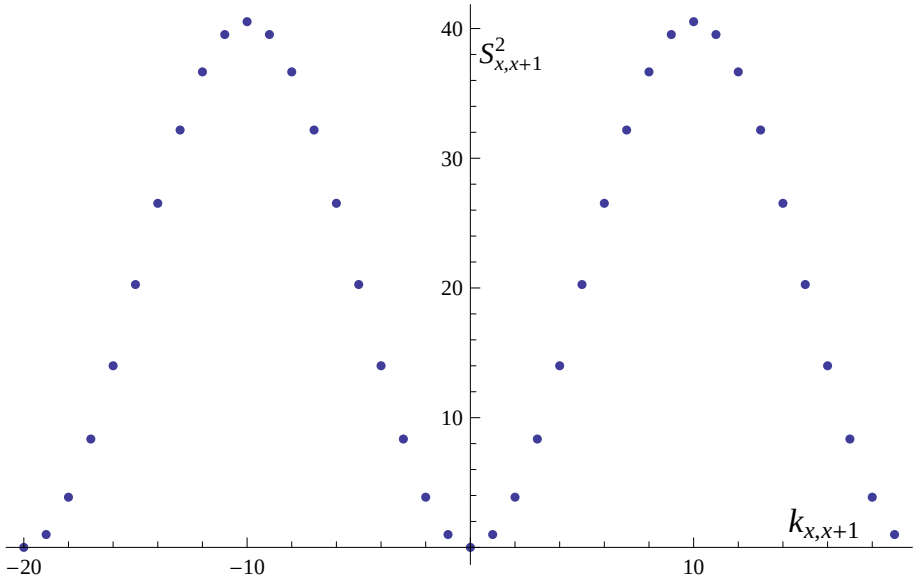

Figure 5.6: The spectrum of $S^2_{x,x+1}$ for $n = 40$.

we obtain the desired result. But inserting in the Hamiltonian a term of the form

$$\frac{g^2}{2} \sum_x S^2_{x,x+1} \tag{5.3.9}$$

would give raise to a degenerate vacuum state since $S^2_{x,x+1}$ is zero for states with $k_{x,x+1} = 0$ or $k_{x,x+1} = -n/2$, which correspond to $v_{x,x+1} = 1$ and $v_{x,x+1} = -1$. In Figure 5.6 the spectrum of $S^2_{x,x+1}$ is displayed.

Conclusions

We studied Abelian gauge theories to define the correspondent lattice model in one spatial dimension. The concepts of fermion doubling and staggered fermions were introduced, and a special attentions was paid to the implementation of gauge transformations in the quantized theory, since gauge invariance provided us with a criterion to select physical states (1.2.33). The lattice model involves a continuous gauge fields which need to be replaced by finite and discrete quantities, in order to define a new model which can be implemented on a quantum simulator.

A widespread model [2, 3, 7] which realizes the discretization of the gauge fields is the Quantum Link Model: we presented an example of its application with a quantum simulator made up of ultracold atoms trapped in an optical lattice.

A critical aspect emerges in the Quantum Link Model: it requires the replacement of a unitary operator, the comparator, with a non unitary one. The $U(1)$ gauge invariance of the model is preserved, and this seems to be enough to define the QLM as a gauge theory; nevertheless, in section (4.1) we have explained the problems which arise from the choice of this model, and we proposed an alternative formalism to define gauge fields on the discrete link Hilbert space.

We obtained that if we define a gauge field on a discrete n-dimensional Hilbert space, the symmetry group of the gauge theory is no longer $U(1)$; the only symmetry group with which a gauge theory can be implemented is the discrete rotation group $\mathbb{Z}_n$. In fact we need to implement the symmetry of the system by replacing the algebra commutation rules with group commutators. We achieved this result by using the formalism of the Schwinger-Weyl discrete group; we first introduced the continuous Weyl group and the Schrödinger representation, and then defined the Schwinger-Weyl discrete group. With these operators, the unitary comparator was defined on lattice links; after that, we defined the gauge transformations with the Schwinger-Weyl's operators, and checked the gauge invariance of the model. By imposing Gauss' law to the vectors of the states Hilbert space we obtained a constraint on physical states, and some difference with the analogous condition in the QLM emerged: states and transitions which are not allowed by the QLM are, instead, not forbidden in our model, even if they physically seem to be counterintuitive. The origin of this problem is the definition of an electric field operator which does not obey the algebra commutation rules, since the symmetry group is discrete and does not admit generators.

Finally, we discussed a possible gauge invariant term to be introduced in the Hamiltonian to represent the energy of the electric field.

The outlook for this work is the definition of suitable observables for the implementation of our model with a quantum simulator: despite the critical aspects emphasized in the text, the QLM is supported by a direct implementation model provided by the Schwinger boson model for the angular momentum [18]. The same should be done for our model: in particular, an observable for the electric field and the comparator should be defined in terms of variables of the quantum simulator. These quantities must yield in the continuum limit the lattice model of Chapter 2, and must reflect the unitary character of the comparator.

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